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# Science of Inexact Mathematics

# Investment Performance Measurement

## Mortgages and Annuities
## Computing Algorithms
## Attribution
## Risk Valuation

AKVY*PRESS

# Chapter 1

# 1. Understanding interest rate and rate of return. IRR equation

## 1.1. Interest rate and rate of return

### 1.1.1. Definition of interest rate

What is an interest rate? There are various definitions that come from economics and finance. The interest rate will be defined for our purposes as *a fraction of borrowed money to be paid to the lender by the borrower as a price for the use of borrowed money*. The actual amount, accrued due to the imposed interest rate, is called *interest*, or interest payment. There are other surcharges and fees associated with loans, such as lender's fee, lawyer's fee and so on, but they should not be considered as part of interest payments. These fees are separate from the interest. Usually, they have to be paid on contract's initiation, at the beginning or before the lending period starts. The other distinguishing feature of interest rate from other fees is that the interest has to be paid *at the end* of lending period, as the cost for the use of money during this period. A lender and a borrower can agree to spread payments over the lending period; other payment options can be introduced as well.

Given this definition, we can write the following simple equation to find the amount to be paid by the borrower at the end of lending period.

$$EV = BV + BV \times r = BV \times (1 + r) \tag{1.1}$$

where $r$ is the interest rate; $EV$ is the ending value, that is the total amount to be paid by the borrower to the lender; it includes principal $BV$ (beginning value) and accrued interest $r \times BV$ .

Suppose, the lender lent money for two periods and expects interest payments at the end of the second period. The interest rate is agreed upon at the beginning of the first period. Interest, which is accumulated during the first period, will have to be added to the principal at the beginning of the second period. During the second period, the interest will be calculated on this total amount. So, the ending value of the first period will be the beginning value of the second period. Substituting formula (1.1), we can find the total amount to be paid at the end of the second period as follows:

$$EV_2 = BV_2 \times (1+r) = BV_1(1+r) \times (1+r) = BV_1(1+r)^2 \quad (1.2)$$

where index ‘2’ relates to the second period.

Similarly, we can add the third period and so on, so that eventually we can generalize that the ending value for the *n*-th period will be the following

$$EV_n = BV_n(1+r)^n \quad (1.3)$$

where $n \geq 1$. However, our assumption is not a mathematical proof yet. To provide a strict mathematical proof, we will use the method of mathematical induction. It implies that some relation is true for particular numbers 1, 2,…. We suggest that it is also true for some number *n*. If we can derive from this assumption that it is also true for the $(n + 1)$-th number, then, according to the principle of mathematical induction, it is true for any number $n \geq 1$. Using this approach and formula (1.3), we can write:

$$EV_{n+1} = EV_n(1+r) = BV_1(1+r)^n(1+r) = BV_1(1+r)^{n+1} \quad (1.4)$$

Hence, formula (1.3) is valid according to the principle of mathematical induction for any number $n \geq 1$.

We can do a very similar exercise when the power is negative. In this case, we should rewrite equation (1.3) in the following form.

$$EV_{n+1} = BV_1(1+r)^{-n}$$

assuming $n \geq 0$. Then, we have to prove that $EV_{n+1} = BV_1(1+r)^{-(n+1)}$. This is done exactly in the same way as in case of (1.4), the only difference is that we have to multiply the *n*-th ending value by $(1+r)^{-1}$.

### 1.1.2. A general note about mathematical approaches

The above proof is mathematically correct. We will pay close attention to mathematical correctness in this book. This is common sense that mathematical statement has to be proved rigorously, because this is what mathematics is about. It is a precise science not in the sense of its numerical results and determinism of its solutions (think about mathematical statistics and probability theory). It is an exact and precise science because of the scrupulous consistency of its methods and the whole conceptual framework.

This framework is built on strict rules and principles preventing origination of conflicting and mutually exclusive constructs within this framework. In this sense, mathematics is a closed system. If one thinks about this, he will realize that closeness is the only way to build the system that does not allow conflicts principally, by definition. It means, by the way, that any new addition has to be non-conflicting to all previous constructs, and this is not an easy task. For instance, mathematical logic in its present state has some built in flaws resulting in conflicting results obtained by this apparatus. So, albeit the word “mathematical” presents, in a strict mathematician’s sense this is not exactly a mathematical discipline.

Some constructs certainly may reside without conflicts in its closed shells, while having conflicting axioms with other systems. For example, Euclid’s and Lobachevsky’s geometries can be viewed as such mutually excluding conceptual constructs. However, in mathematics, if some constructs intersect, they cannot conflict. This is one of the major concept and beauty of mathematics, which creates the world of non-conflicting abstract constructs. However, many of them describe phenomena of a real life quite adequately. This may be seen as a paradox because the real life is much more complicated, and it principally has numerous inherent contradictions and conflicts. However, there is an answer to this question too, which is based on approximation and iteration notions. In such situations, we model a certain side, or aspect of the reality through approximations and iterations, because we have a powerful tool for doing this adequately. This tool can be considered as sort of two way communication channel with some verification and validation tools on both ends, through which our reasoning abilities do dialog with the reality, and this wonderful and absolutely indispensable tool is called *practice*.

## 1.2. Introducing compounding

Suppose some lender does not want to subdivide an entire period into two smaller time intervals. What interest rate is it necessary to charge to receive the same interest payment? This requirement can be represented using equation (1.2) as follows.

$$BV_1 \times (1 + r_1)^2 = BV_1 \times (1 + r_2)$$

where $r_1$ corresponds to interest rate for a shorter period, and $r_2$ corresponds to an interest rate for longer period (the one composed of two smaller periods).

Solving this equation with regard to $r_2$, we find

$$r_2 = (1 + r_1)^2 - 1 = r_1 \times (r_1 + 2)$$

For example, if $r_1 = 10\%$, then $r_2 = 21\%$, but not 20 %. This is where we encounter the concept of *compounding*. The difference between two scenarios is this. In the case of shorter time periods, we calculated accrued interest, and added it to the principal in the middle of the lending period. When the second period started, its beginning value was greater than the beginning value of the first period, by the amount of interest that was accrued in the first period.

This operation is equivalent to making the principle bigger in the second period. In case of a single longer period, the principle did not change, and the accrued interest was calculated only at the end. This is why the lender has to charge a higher interest rate, when he lends money for a single period, in order to get the same interest payment as he receives in case of lending money for two periods. Obviously, the more periods we have, the more the difference is noticeable. For example, if five periods are involved with 10 % interest rate in each period, then the equivalent interest rate for a single period, which is composed of all five periods, is 61.05 %, but not 50%.
So, as long as we deal with multiple periods and use formula (1.3) or its derivatives, compounding is always implicitly involved.

If we do not want to compound the interest, then we should use a different approach. In this case, each period has to start with *the same* principal amount. We can write an equation for $n$ periods as follows.

$$EV = BV + n \times (BV \times r) = BV \times (1 + nr) \tag{1.5}$$

It is important to understand the issue of compounding and use formulas consistently. If one mixes compounding and non-compounding approaches together (or contexts, as we will call them in this book too), such creativity produces invalid results, and sometimes it brings an issue to the court rooms. We will discuss a few examples later that demonstrate typical misunderstandings on compounding and non-compounding issues.

We will consider compounding in more detail throughout the book, but it is important to begin analyzing this phenomenon as early as possible, because compounding is one of the cornerstones of finance; it is intensively used in investment performance measurement and accounting. Compounding often invisibly presents in the tissue of mathematical methods employed in these disciplines. So, it always important to know the right problem's context, that is compounding or non-compounding.

# 1.3. Domain of interest rate. Range of applicability

### 1.3.1. Specifics of negative interest rate

In the previously introduced formulas, the interest rate can be negative from mathematical perspective. Let us construct a business case. Suppose there is a situation when nobody wants to borrow money because of the looming monetary reform, which will happen at unknown time, perhaps in a year or two. Old banknotes will not be exchanged and the government will not denominate the currency. In such a situation, it makes sense to lend money for a period of two years at a negative interest rate in order to have the opportunity to receive a fraction of old money in new banknotes. Suppose the lender and the borrower agree on interest rate of (-8 %), with principal being \$100. In two years the lender should receive

$$\$100 \times (1 + (-0.08))^2 = \$100 \times (0.92)^2 = \$84.64$$

which is better than nothing. The parties may agree on (-50 %) of interest rate. They can agree even on (-100 %), so that the lender will legally loose all his money and then file bankruptcy. We do not know all possible scenarios. However, life is inexhaustible in these sorts of achievements. So, there will always be cases which we cannot anticipate, including the ones that lead to anomalies like (-100%) of interest rate.

Let us continue to explore the domain of interest rate. Can interest rate be (-150 %)? If we assume this, then the base becomes negative. It is mathematically legal to raise any negative number to negative power if the power is an integer number. So, the power in this case can be negative. For instance, $(-1.1)^{-3} = -0.1079797$. However, it is illegal to raise a negative real base to a negative real power.

Do we need a real negative power? We should say yes thinking from the business perspective. Imagine the following situation. The lender wants to have an option to get all his principal and accrued interest before the expiration of a lending period. If the lending period was four years, and the lender decided to exercise this option after three years, the borrower has to pay the following ending value:

$EV = BV \times (1+r)^{0.75}$, because the actual lending time is equal to three quarters of the original time period. So, the power should be allowed to be a real number from the business perspective. Then, we should revisit formula (1.3). It has been derived with the assumption of integer non-negative power. Generalization of formula (1.3) for the real power is a consequence of additive property of powers when the base is the same. That is:

$$B^d = B^{a+b+c} = B^a B^b B^c \text{, if } d = a+b+c .$$

We can rewrite the numerical example above using this additive feature as follows,

$$EV = BV \times (1+r) = BV \times (1+r)^{0.75} \times (1+r)^{0.25}$$

dividing the whole period into two parts - three quarters and one quarter. Value $EV = BV \times (1+r)^{0.75}$ corresponds to the ending value, which includes principal and interest to be paid to the lender after three quarters of the total period.

If the power is real, then the base cannot be negative according to mathematical constraints introduced above. So, the interest rate $r \geq -1$. In business terms, it means that a lender cannot loose more than the beginning value. Formula (1.3) is an exponential function if we assume that *n* is a *real* number. Mathematically, the base can be any non-negative number, while the power can be any real number. So, $-1 \leq r < \infty$, $-\infty \leq n < \infty$. The upper limit in the last inequality does not interfere with business restrictions. The number of periods can be made larger and larger, as well as the interest rate.

Can the power be negative from the business perspective? What would be a business situation when the power becomes negative? Suppose some lender just lent \$100 for two years at interest rate $r = 12\%$. He asks himself a question, if he lent money a year ago, how much he would have to lend at that time in order to expect the same interest amount in two years? The answer is:

$$EV = BV \times (1+r)^{-1} = \$100 \times (1+0.12)^{-1} \approx \$89.29$$

What if he would have to lend money three and a half years ago? The answer is as follows:

$$EV = BV \times (1+r)^{-3.5} = \$100 \times (1+0.12)^{-3.5} \approx \$67.26$$

So, we can calculate the ending value not only in the future direction, but also into the past. It is a tradition that time increases in the direction of future. So, some people will not agree with our innovative proposal to call the preceding event as the ending event. This is why a special terminology has been invented not to mismatch the future with the past. When we compute an ending value toward the past, this value is called as a *present value*, to distinguish it from the *future value*, which is the ending value somewhere in the vague future.

The operation of computing the present value (backward computing of the ending value) is called *discounting*. Business meaning of this operation can be interpreted as if someone invested money at different moments of time in the past in such a way that the resulting future value will be the same at some fixed time. This discounting operation is used in different financial calculations. For example, for calculating implied volatility in the option pricing models.

### 1.3.2. The meaning of compounding when the period length is not an integer value

Some people may not be comfortable that we increased the domain of time periods replacing an integer power by a real number. The idea of compounding revolves around adding the interest, accrued during the previous period, to the portfolio value at the end of this period. This is how we introduced accumulation of accrued interest within the portfolio. If the power is not an integer, what then? How can one add the accrued interest inside the period?

Mathematically it is possible, we demonstrated this. However, what about the practical application of this approach? It turns out there are objections to this approach, and some compromise is required. For example, on the period boundaries, we use compounding in its classic form, while in between those boundary points we calculate the accrued interest based on a non-compounding scenario. We will discuss this question later in detail. For now, we would like to pay attention that the issue is not as straightforward and requires more study. The table below summarizes these considerations.

Table 1.1. Mathematical constraints applied to calculating interest rate and their business consequences

| Mathematical Constraints | Business Consequence |
|---|---|
| The base cannot be negative for real power, so the minimum interest rate $R \geq -1.0$. | The lender cannot loose more than the beginning investment value. |
| The power can be any real number | It is possible to calculate the ending value in the future and in the past directions. In the first case, this number is called a *future value*, in the second case it is called a *present value*. |

# 1.4. Computing interest rate

## 1.4.1. Relationship between the period length and interest rate

We will rewrite (1.3) as follows.

$$EV(T) = BV \times (1+r)^T \qquad (1.6)$$

This notation reflects the fact that a period length can be any real number, and the ending value is a function of period length.

The consequences of this substitution are not as trivial as it seems at first. It ties together an interest rate and units of measure for the period. The unit of measure for a period with length $T$ is a period with one unit length, to which we apply the interest rate. If $r$ is the *annual* interest rate, then $T$ has to be measured in *years*. Otherwise, the result will be invalid. If some lender applies weekly interest rate of $20\%$, and lends $100,000, then the ending value to be repaid in a week is $\$100{,}000 \times (1+0.2)^1 = \$120{,}000$. However, if analyst mistakenly measures the period in days, then the calculation produces $\$100{,}000 \times (1+0.2)^7 = \$429{,}980$. This is an invalid result. In fact, it might be the disastrous one.


*So, the lending period has to be measured in units of time the interest rate is applied to.*

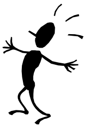
If this is a weekly interest rate, and the lending period is three weeks, then we should substitute $T = 3$ into equation (1.6). If this is a monthly interest rate, and the month has 30 days, then $T = (3 \times 7)/30 = 0.7$. If the month has 28 days, then $T = (3 \times 7)/28 = 0.75$.

## 1.4.2. Computing interest rate for shorter or longer periods. Nominal and effective interest rates

The next question is, how to compute an interest rate for some period, provided we know the interest rate for a period with different length. If the annual interest rate is 120 %, would it be valid to assume that the interest rate for a quarter is (120)/4=30%? What kind of method should we use? This is where we have to take into account an application context.

Do we want to use the obtained interest rate in a compounding scenario and henceforth to use formula (1.3)? Or, are we going to ignore compounding, and consequently use the non-compounding context and hence formula (1.5)? This is not a hypothetical but, indeed, a practical situation. A financial analyst has to make this choice every day (unless software does this for him). Presently, these issues are allegedly resolved by introduction of certain artificial constructs. In particular, the *nominal interest rate* and *effective interest rate* are such constructs. In the example above, the *nominal* annual interest rate is 120 %. Then, the monthly interest rate will be calculated as 120/12(months) = 10%. (If thus obtained monthly interest rate should be called a nominal interest rate, or somehow else, depends on the following usage, but presently this question is ignored, in order not to add more ambiguity.) Anyway, such a monthly interest rate produces the *effective* annual interest rate as follows: $(1+0.1)^{12} = 213.8\%$, which is very different from the original 120 %. This is because compounding, assumed in this case *implicitly*, enters to the scene.

Without compounding, the nominal interest rate becomes effective interest rate, which is also equal to 120 %, when we apply similarly the back and forth transformations ($120/12 = 10\%$, $10 \times 12 = 120\%$).



This approach looks complicated. However, the proposed explicit introduction of the notion of compounding context helps to clarify the issue, but only to some extent. According to this approach, in addition to these non-intuitive terms we have to add more definitions and certain conditional phrases. Then, the computed value for the effective annual interest rate supposed to become more legitimate. Namely, we have to say exactly this: “Nominal annual interest rate at 120 % compounded monthly”. This phrase is still cryptic, but at least compounding is mentioned this time. This is why financial analysts have to choose the words carefully when explaining what kind of interest rate is discussed. However, the problem is that the general public lacks this refined knowledge, while these people always present on the other side of lending equation.

In fact, this simple issue of computing interest rate has been overcomplicated without good reasons. There is only one interest rate that can be simply and unambiguously converted to an interest rate for a longer or shorter period. We just have to specify the compounding or non-compounding context. Overwhelming majority of practical applications, such as mortgages and annuities, assume a compounding context.

There are no mathematical or business reasons to introduce the nominal and effective interest rates, as well as a complicated conditional wording for their manipulation. The problem is that majority of users do not understand, or quickly forget, these intricacies and simply begin to divide the annual interest by the number of months, if they compute a monthly interest rate from the annual interest rate. When they need semiannual interest rate and know the monthly interest rate, they multiply the monthly interest rate by six. No reservations, no conditional words, no mentioning effective or nominal interest rates. This is how everyday practice "corrected" unnatural constructs, which, in fact, mangle the original idea and create invalid approach. One can go to the Internet and find thousands of such erroneous examples.

### 1.4.3. Mathematical foundations of interest rate calculations

Let us to consider an example. Suppose we want to do quarterly compounding correctly from the perspective of nominal and effective interest rates. What do we have to do if we know an annual effective interest rate? We should not divide it by four, should we? Should we find a power of $1/4$ of this number? Apparently yes, but we are not sure if the lender would agree with this interpretation given the following consequences.

The annual nominal interest rate is 120 %, as before. The first approach is to use compounding and find a quarterly interest rate as $(1+1.2)^{1/4}-1\approx 0.2179 = 21.79\%$. With the second approach, when we apply the notions of nominal and effective interest rates, we have to calculate the ending value for a quarter using the original 120%, and applying a simple dividing rule, which produces interest rate 30 % for the quarter, effectively creating a non-compounding application context. However, in today's practice, this context is usually not mentioned.

Let us compare the results. The beginning value is $100. In the first case, $EV_1 = \$100 \times (1+0.2179) = \$121.79$. The second approach produces $EV_1 = \$100 \times (1+0.3) = \$130$. If we do calculation for one year period, then we have $EV_1 = \$100 \times (1+0.2179)^4 \approx \$220$ and $EV_1 = \$100 \times (1+0.3)^4 \approx \$285.61$ respectively. The results are substantially different. We have no doubt with regard to the first compounding approach, because we did calculations from scratch according to derived formulas. So, the problem is with the second method.

Mathematical consideration of this phenomenon is as follows. If the second method is true, then the following equality to be held.

$$(1+r)^{\frac{1}{T}} = (1+\frac{r}{T}) \qquad (1.7)$$

The following transformations can be done. We raise both sides of equation (1.7) to power *T*. Both sides are positive. So, this operation is an equivalent mathematical transformation. We obtain:

$$(1+r) = (1+\frac{r}{T})^T \qquad (1.8)$$

Right side of (1.8) is a binomial sequence. We can rewrite it as follows (Salas, 2007).

$$1+r = 1+r+\frac{T(T-1)r^2}{1 \cdot 2 \cdot T^2}+\frac{T(T-1)(T-2)r^3}{1 \cdot 2 \cdot 3 \cdot T^3}+..... \qquad (1.9)$$

Formula (1.9) shows that the left and right sides of equation are not equal. If *T* is an integer, then the sequence on the right side has a finite number of terms, all of them are positive. If *T* is not an integer, the number of terms is infinite. So, our assumption is invalid, and, consequently, the method itself is invalid. Nonetheless, this method has a wide acceptance in this industry, and it is used in numerous compounding calculations, while it works *only* for non-compounding scenarios based on formula (1.5) and its variations.

Mixing different contexts in the form of effective and nominal interest rates, and adding conditional phrases to resolve this inherent conflict between different contexts, does not work. And it should not, because the normal human mind is not the right place for storage of conflicts and controversies. It will cut out conflicting things and leaves something non-contradictive, but not necessarily right. The best solution would be to stop using the nominal interest rate, and begin to use the interest rate as a single notion. In addition, the application context should be defined, which is presently the compounding context almost without exception. In this case, recalculation of interest rate to shorter or longer periods becomes a simple, straightforward, and unambiguous procedure, which everybody can understand and use correctly.

However, such useful improvement will touch many sensitive strings. The real problem is that the mental inertia of acquired knowledge is proportional to the exponent of weight of people who accepted some paradigm. This is why when one goes through the literature and manuals, or internet pages, he will not encounter explanations and distinctions made between the nominal and effective interest rates but, instead, he will find just an interest rate that is used without distinguishing across the boundaries of compounding and non-compounding contexts, as if this is a single territory. This is how people responded to overcomplicated constructs representing the notion of interest rate.

When one wants to invent some new construct for the general use, he should think far beyond his own capability to hold this construct in his mind for short period of time. At this design time, he is totally involved into the creation process. Most likely, the designer himself will not understand what he created after several months after the project completion. (We see such situations in software development projects on a regular basis.) So, the creator should not have grudge against the ordinary user of his method.

*Within a compounding context, the approach to obtain an interest rate for a shorter period as a proportion of the interest rate for a longer period mathematically is invalid.*

### 1.4.4. Computing interest rates. Numerical examples

Below we provide a numerical example for a smaller value of interest rate, in order to see, how critical is the mixture of different contexts in this case. Let us assume $r = 5\%$. Then, the compounding approach delivers the result

$$EV = \$100 \times (1 + 0.05)^{1/4} = 101.227$$

The second, proportional approach produces

$$EV = \$100 \times (1 + 0.0125) = 101.25$$

The difference is 1.3 cents, which doesn't look as a big variation. The problem is that this is a systematic error. If we use invalid interest rate to calculate the ending value for four periods in the compounding scenario, then we will get the difference of 9.5 cents on a sum of $100. Eight periods produce 20 cents. In practice, the number of periods can be tens and hundreds, which is a common situation with mortgages, and transaction values much bigger than our paltry $100. So, this can be noticeable amount when it is accumulated across multiple periods and/or across multiple financial instruments. The error grows rapidly with the increase of interest rate, far quicker than linear dependencies. This fact is illustrated by Table 1.2 and Fig. 1.1.

The first method assumes a compounding context, while the second method uses an effective interest rate and a nominal interest rate for computing interest rate for smaller periods.

So, analysts have to exercise consistent approach when manipulating interest rates. This consistency presumes to remain within the boundaries of compounding or non-compounding contexts, and not to cross the border line between them. Otherwise, the results will be invalid.

Table 1.1. Differences in interests produced by two methods for the same principal.

| Interest Rate, % | Interest, $ difference for one period per $100, between the compounding and nominal interest rate methods | Difference, % accrued for the total period, composed of four periods | Difference, % accrued for eight smaller periods |
|---|---|---|---|
| 5 | 0.023 | 0.095 | 0.2 |
| 50 | 1.83 | 10.18 | 31.6 |
| 80 | 4.18 | 27.36 | 106.0 |
| 120 | 8.21 | 65.61 | 331,7 |

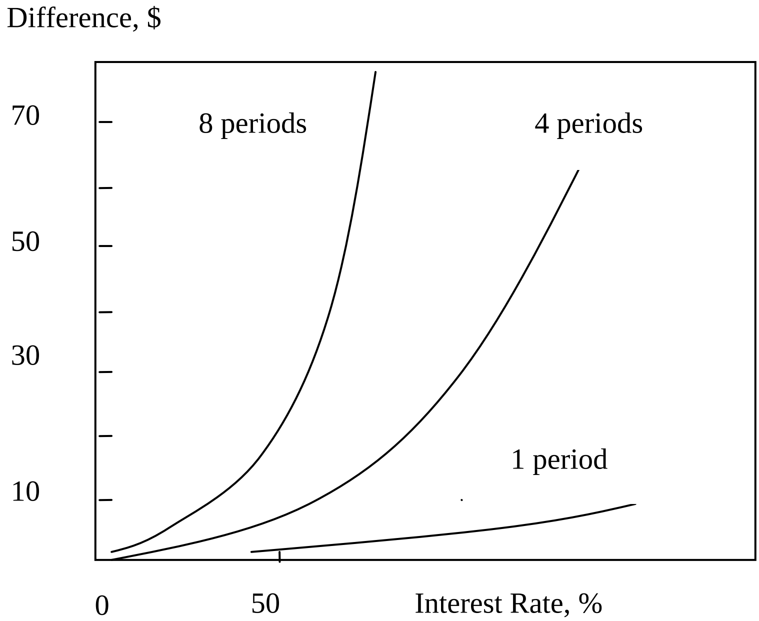



Fig. 1.1. Difference between the interests computed by two methods.

## 1.5. Continuous compounding.

The previous section convinced us that we should know the context of the problem, which is whether compounding or non-compounding. There is a possibility to merge them together, but this has to be done cautiously and rightly, and we will present such a case later. An attempt to mix these concepts through the nominal and effective interest rates is not a success story, as we have seen already. However, this approach is still in use and the reader should understand its specific. In this section, we will enhance the understanding of compounding context thoroughly studying its properties.

Previously, we considered discrete periods only. However, compounding is a very beneficial option for the lender. It makes sense to have as many periods as possible, ideally infinitely small and countless. (Human nature often pushes boundaries to their limits and beyond, until it is too late.) Mathematics, in particular calculus, supports these human psychological virtues and provides appropriate quantitative instruments.

### 1.5.1. Continuous compounding. Definitions

We will do the derivation of formula for continuous compounding of interest. Continuous compounding means the following. Imagine an extremely short time interval $dT$ within a fixed period with length $T$. Some interest will be accumulated at the end of this short interval. This interest is immediately added to the total value of the portfolio, so that at the beginning of the next short period with the same length $dT$ the beginning value is increased by this interest amount. The periods become shorter and shorter, approaching infinitely small value, while preserving the total period length $T$. The limit of this procedure, when the period length goes to zero, produces continuous compounding.

First we consider a simple scenario when a long period is equal to one. We will start from a finite number of periods, and then find the limit of the expression when the size of the smaller periods approaches zero. Let $R$ be the interest rate for the total period, $r$ is the interest rate for a short period. We should use formula (1.3) for compounding context, in order to find the interest rate for this short period. Equating the ending market values for the single and multiple periods, we will obtain the following expression.

$$(1+R)=(1+r)^n \qquad (1.10)$$

Solving this equation, we will find the interest rate for the short period.

$$r=(1+R)^{\frac{1}{n}}-1 \qquad (1.11)$$

So, if we compound interests for such small periods, then the ending value is defined by the following formula.

$$EV=BV\times[1+r]^n=BV\times\left[1+((1+R)^{\frac{1}{n}}-1)\right]^n \qquad (1.12)$$

This is a correct compounding formula for the finite number of periods. Formula (1.12) readily transforms to familiar equation $EV=BV\times(1+R)$ when $n$=1. This is one of the confirmations that our results are valid, according to the theory of verification of scientific knowledge that considers continuity of transformations on the domain boundaries.

Now, we have to find the limit in the right side of expression (1.12), when $n$ goes to infinity. We can use the same approach as in formulas (1.8) and (1.9) representing the term $(1+R)^{\frac{1}{n}}$ as a binomial sequence.

$$(1+R)^{\frac{1}{n}}=1+\frac{R}{n}+\frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\times R^2}{2!}+\ldots+\frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\ldots\left(\frac{1}{n}-k\right)\times R^k}{k!}+\ldots$$

(1.13)

Substituting (1.13) into (1.12), we will obtain the following equation.

$$EMV = BMV \times \left[ 1 + \frac{R}{n} + \frac{\frac{1}{n}\left(\frac{1}{n} - 1\right) \times R^2}{2!} + \ldots \right]^n \qquad (1.14)$$

The limit of the right side in (1.14), when *n* goes to infinity, is found as follows.

$$\lim_{n\to\infty} \left( 1 + \frac{R}{n} + \frac{\frac{1}{n}\left(\frac{1}{n} - 1\right) \times R^2}{2!} + \ldots \right)^n = \lim_{n\to\infty} \left( 1 + \frac{R}{n} \right)^n =$$

$$= \lim_{n\to\infty} \left( 1 + \frac{R}{n} \right)^{\left(\frac{n}{R}\right)R} = \lim_{p\to\infty} \left( 1 + \frac{1}{p} \right)^{(p)R} = e^R \qquad (1.15)$$

where $p = n/R$.

Here, we use properties of function limits, and the a known limit that can be found in (Salas, 2007).

$$\lim_{p\to\infty} \left( 1 + \frac{1}{p} \right)^p = e$$

So, we can rewrite equation (1.14) as $EV = BV \times e^R$. Now, we should generalize this formula. We assume that a longer period has the length *T*. This length has to be measured as the number of periods to which the interest rate *R* is applied. That is, if the interest rate corresponds to a three months period, and we would like to calculate the compounded ending value for 15 months, then $T = 15/3 = 5$. We should always remember about the inherent relationship of period lengths and corresponding interest rates. Equation (1.12) can be rewritten as follows.

$$EV = BV \times [1 + r]^{nT} = BV \times \left[ 1 + ((1 + R)^{\frac{1}{n}} - 1) \right]^{nT} \qquad (1.16)$$

The only difference is that in this case we calculate compounding for different number of periods, which is *nT*, and *T* is a real number, not only an integer. Except for the outer power, the rest of transformations is exactly the same as in formulas (1.13) – (1.15). The final result is the following.

$$EV = BV \times e^{RT} \qquad (1.17)$$

Note that values *T* and *R* inherently relate to each other. We can rewrite this equation as follows to emphasize this relationship.

$$EV = BV \times \exp(R \times (T / t_R))$$

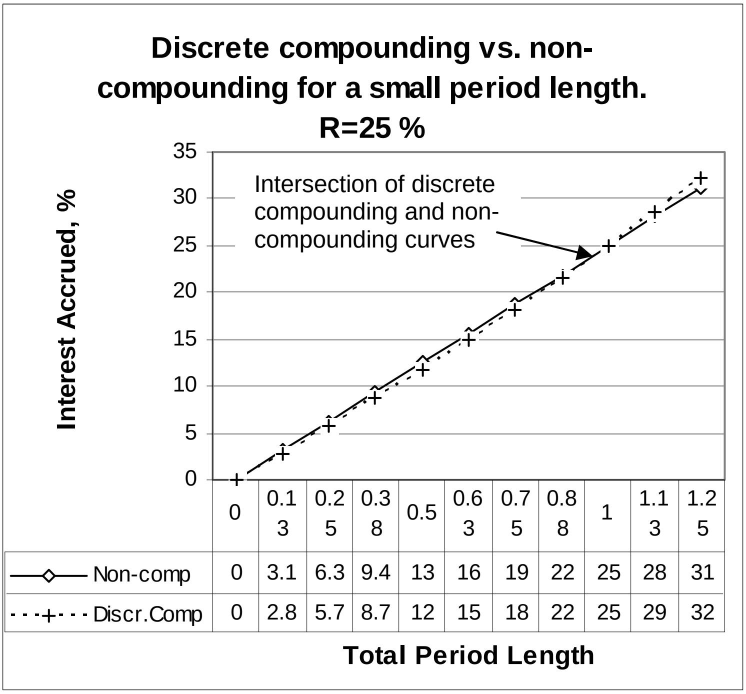
where $t_{R,}$ is the length of the period to which the interest rate *R* is applied, and both periods are measured in the same units of time.

### 1.5.2. Continuous compounding versus discrete compounding. Numerical example

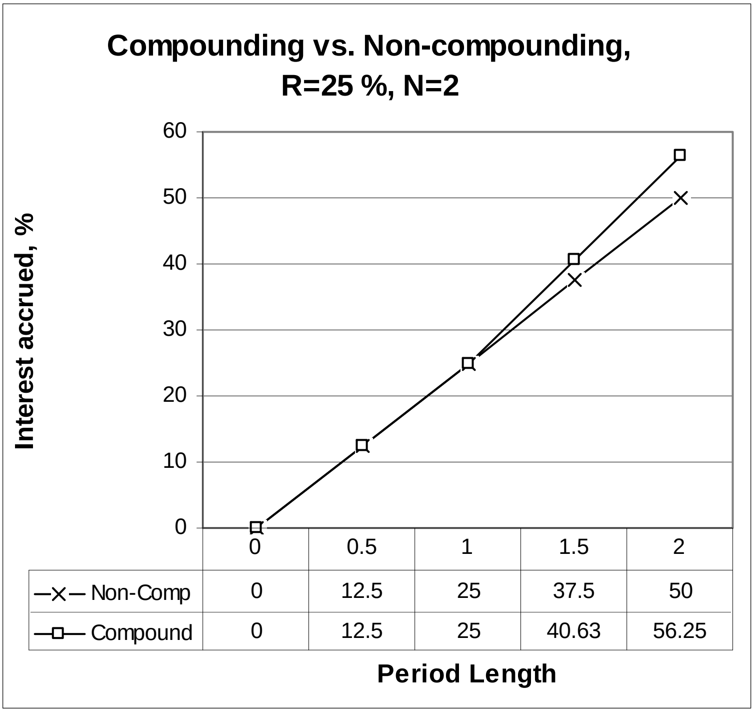
We will provide numerical examples in this section. First, we consider compounding and non-compounding dependencies near the point $T = 1$. If we apply formulas (1.5) and (1.6) directly, we will get the graphs and tables shown in Fig. 1.2-a. The compounding curve is *below* the straight line representing the discrete compounding. In fact, the straight line represents the non-compounding scenario in the range $0 < T < 1$. When $T > 1$, the curves swap their positions, and the compounding curve climbs *above* the non-compounding line. Both graphs intersect at point $T = 1$. (The existence of this intersection point can be proven analytically using calculus theorem about intersecting of functions with different first derivatives in a given range. We will not provide the proof here. It can be found in many textbooks such as (Salas, 2007), or the reader can do it himself.)

It is important to pay attention that in Fig. 1.2-a the graph of a compounding function goes *below* the non-compounding dependence.

This is the actual behavior the formulas provide. However, in business practice, these graphs are treated as coinciding in the first period. To some extent, this is done for convenience and simplification of mathematical calculations. Note, if calculations are done correctly, then the interest to be paid will be less. Nonetheless, it is assumed in business practice that compounding and non-compounding curves coincide in the first period, which is illustrated by Fig. 1.2-b.




a)




b)

Fig. 1.1. Accrued interest as a percentage of the beginning value of \$100, for compounding and non-compounding scenarios. a) behavior of graphs near the point $T = 1$. b) graph of the actual compounding function used in business practices, versus the non-compounding scenario. Interest rate $R = 25$ %, number of periods $N = 2$.

### 1.5.3. Smooth exponential compounding function versus the piecewise linear function

Compounding method defined by formula (1.6), and the appropriate graphs in Fig. 1.2-a, are mathematically correct. Smooth exponential functions provide adequate quantitative description of the compounding phenomenon for any real power. However, we noted that the current business practice is different from what is prescribed by pure mathematics. There are arguments from the business perspective that the compounding graph in Fig. 1.2-a is not adequate. Compounding can be done only at certain discrete points because this is how the discrete compounding is defined. Between these points, the interest growth is approximated by a non-compounding approach. This is demonstrated by graphs in Fig.1.2-b, so that the "practical" compounding curve is a piecewise linear function.

The curve, which is defined by compounding formula (1.6), coincides with this piecewise linear function at cusp points. Is this the right approach? Given the available inputs, we can say that both approaches can be used. From the mathematical perspective, nothing is wrong with either method. To a large extent, this is a matter of agreement. The lender is in a better position with a piecewise linear function, when the loan has to be repaid between the cusp points, which are the boundaries of periods. In this case, an accrued interest is slightly higher. However, this may not be the main reason for the introduction of this approach.

So, we have a classic problem of choosing between two approaches. The first question is, do we really have to make a choice? Consistency and non-contradictoriness of methods is very important for this industry. The mere existence of two different approaches within the same area creates an ambiguity, because then people will choose method subjectively.

Mathematics is an instrument. This is a tool applied presently by humans to real problems to find a quantitative solution. Mathematics does not produce solution on itself, even if one strictly follows its rules. People's interests can interfere with a decision making process at any phase. This intervention not only can tip the scale, but bend the outcome to opposite direction when necessary. For instance, statistics has more than enough degrees of freedom to deliver the desired result. The same consideration is valid for any other phenomenon people engage to, being it another branch of natural science, technology or economics. So, we have to include other considerations beside the pure mathematical algorithms when resolving a real life problem, in order not to loose the adequacy of our presentation and objectivity.

In our case, we cannot get rid of exponential function, because it is present in compounding by definition. So, it has to be incorporated into the chosen method. This is the first argument.

The second consideration relates to flexibility in period composition. Usage of smooth exponential functions allows creating of arbitrary combinations of periods, including periods with different lengths. This is a very beneficial feature for analytical studies. Piecewise linear function is more rigid instrument with regard to such functionality. Every time we change the period length, we have to change the linear function between the cusp points, which significantly reduces the method's generality and practicality. So, the smooth exponential function provides better flexibility in period composition, while the piecewise linear function does not.

The third argument relates to the IRR equation itself. We would like to point out that the IRR equation is based on smooth exponential function, not a piecewise linear function.

The piecewise linear function can be incorporated into IRR equation, which will lead to its mathematical modification. This equation is more complicated than the original IRR equation, both in appearance and transformations.

A piecewise linear approach has advantage from the perspective of doing compounding only at discrete points. However, thus we reduce the generality of approach. For instance, paying debts at arbitrary time is a real business scenario. As soon as we assume that $T$ is a real number in formula (1.6), we have to stipulate that it also describes the accrued interest between the points of compounding, corresponding to integer period length. Such assumption is a very natural one, but we have to explore its consequences. Otherwise, we con change the range of the independent parameter $T$, but leave its domain unchanged. We have to match the range and domain of mathematical formulas. The smooth exponential function enforces this notion complying with the compounding context. So, this is the forth argument in favor of smooth exponential function. So, overall the smooth exponential function is preferable over the linear piecewise function.

All graphs representing an accrued interest in case of discrete compounding with piecewise linear functions are similar. They are piecewise linear functions with cusps on the boundaries of periods, although it may not be seen on the graphs because of the small difference between the slants of neighboring line sections. The difference in slope is caused by adding the interest, which is accrued during the previous period, to the beginning value of the next period. Effectively, this addition can be interpreted as if the interest rate increases for each following period. Let us to prove this.

We consider compounding for two periods, when each period has the length $T$. In the first period, the ending value $EV_1$ is defined as follows:

$$EV_1 = BV_1 \times (1 + T \times R)$$

where $T$ is the period length.

Interest $I_1$, accrued at time $t_1$ ($0 \le t_1 \le T$) in the first period is

$$I_1(t_1) = EV_1(t_1) - BV_1 = BV_1 \times (1 + t_1 R) - BV_1 = [BV_1 \times R] \times t_1$$

This is a linear function of $t_1$ with a slope $BV_1 \times R$.

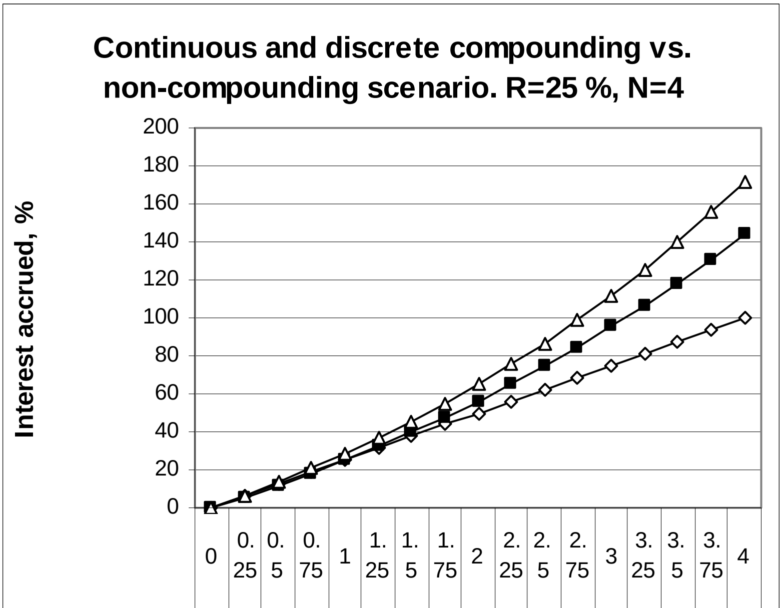
The beginning market value of the second period is the ending market value of the first period, that is $BV_2 = BV_1 \times (1 + T \times R)$. Accordingly,

$$EV_2 = BV_2 \times (1 + T \times R) = BV_2 + BV_2 \times T \times R$$

Then, the interest $I_2$ accrued in the second period at time $t_2$ ($T \le t_2 \le 2T$) is equal to:

$$I_2(t_2) = EV_2(t_2) - BV_2 = BV_2 \times t_2 \times R = [BV_1 \times (1 + TR) \times R] \times t_2$$

This equation defines the linear function $I_2(t_2)$ of argument $t_2$ with a slope $[BV_1 \times (1 + TR) \times R]$, while the first linear piece has a smaller slope $[BV_1 \times R]$. The slope for the second period is steeper than the slope of the linear function corresponding to the first period. This way, the piecewise linear function, that describes compounding, climbs steeper and steeper for each following period. When the number of periods is increasing for the same total period, this function approaches exponential function corresponding to continuous compounding.




| | 0 | 0.25 | 0.5 | 0.75 | 1 | 1.25 | 1.5 | 1.75 | 2 | 2.25 | 2.5 | 2.75 | 3 | 3.25 | 3.5 | 3.75 | 4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Non-comp | 0 | 6 | 13 | 19 | 25 | 31 | 38 | 44 | 50 | 56 | 63 | 69 | 75 | 81 | 88 | 94 | 10 |
| Discr.Comp | 0 | 6 | 12 | 18 | 25 | 32 | 40 | 48 | 56 | 65 | 75 | 85 | 95 | 10 | 11 | 13 | 14 |
| Cont.Comp | 0 | 6 | 13 | 21 | 28 | 37 | 45 | 55 | 65 | 76 | 87 | 99 | 11 | 12 | 14 | 15 | 17 |

Fig. 1.3. Accrued interest as a percentage of beginning value for non-compounding, discrete compounding and continuous compounding scenarios, for four periods. $R$ = 25 %, $N$ = 4.

Fig. 1.3 illustrates these features of compounding operation. The graph shows discrete compounding, which consists of four linear sections. More frequent compounding leads to a higher value of accrued interest.

Completing this section, we would like to reiterate once more on the main topic of the discussion. Compounding can be thought of as an operation when the ending value of the previous period immediately becomes the beginning value of the next period. The total return for multiple periods is defined by equation (1.6) and its derivatives. This approach is *the only* legitimate way to consider compounding related tasks. There are no other alternatives, as we proved in our derivations.

We considered in detail the calculation of interest rate for a period composed of multiple shorter periods. We demonstrated also how the intention to introduce the nominal and effective interest rates, and complicated wording instead of one universal unambiguous interest rate, led to ambiguities in practical applications. It is possible to see in the literature recommendations to divide the total period return by the number of periods, in order to find the rate of return for a smaller period; and these recommendations are introduced without any considerations for the nominal or effective interest rates. We proved that this is valid approach for *non*-compounding context, but the problem is that the actual lending business is done within the *compounding* context almost without exceptions.

Introduction of different interest rates within the same context (either compounding or non-compounding) is unnecessarily overcomplicated approach that ignores the very fundamentals of the compounding operation. Every quantitative instrument has its domain of applicability, and has to be used only within this domain. Compounding problems have to be investigated using compounding related quantitative approaches, while introduction of nominal and effective interest rates brings confusion into this industry. The reason is that the approach tries to tie together two incompatible things, compounding and non-compounding contexts. It is unlikely that this practice will go away soon, but at least the users should understand the implications and adapt to them knowingly.

## 1.6. IRR equation. An inherent relationship of compounding operation and cash flows

It is a long story until we will introduce a hierarchy of different methods for calculating rate of return. We will show in the next chapters that internal rate of return (IRR) is the main method, from which other methods for calculating rate of return are derived. If one is looking for the most objective parameter to characterize the return on an investment portfolio, this is most likely to be the IRR, or its direct sibling. If one wants to develop the most adequate model of investment scenario, then IRR most likely to be included into this model, in order to insure the best possible adequacy of the model to a real phenomenon.

The present status of IRR method is respectable already, and the method is gaining more esteem. However, the IRR method is considered presently in the investment industry as one method in the set of others that accompanies independent and equally important methods, while the truth is that IRR is the parent of other methods, in the overwhelming number of investment performance measurement scenarios.

Until recently, each method in this set used to be considered independent. In reality, every phenomenon has some structure and inner dependencies; there are no absolutely amorphic entities. The parameters defining the phenomenon are interconnected not in an arbitrary fashion, but in some structure. Otherwise, the phenomenon cannot exist. Nobody can deal with something shapeless, amorphous and intangible. The phenomenon itself does not exist in a vacuum, but it interconnects, in one way or another, to other reality phenomena. If each of methods, which characterize the same phenomenon from the same perspective, produces different value, which is in many instances exactly the case with the performance measurement methods, then, if not an alarm, but the question to be raised with regard to the adequacy of these methods.

One common opinion within the investment performance specialists, which still has adherents, is this. There are many rates of return, each servicing some particular specific need. For example, the investor is interested in a total return, while the fund manager's performance has to be evaluated differently, because he has no control over the cash flows and transactions initiated by investor. The time weighted rate of return (TWRR) is supposed to fulfill this requirement by "purifying" the rate of return from cash transactions. We will show in the appropriate chapters that TWRR is an inadequate measure of rate of return in many instances. Its analysis should be done with relation to other methods, in order to understand its drawbacks. There are no absolute truths and TWRR is not exception.

On the opposite side, IRR takes into account all cash flows and transactions objectively, but one condition has to be fulfilled – the problem has to be considered within the compounding context. Normally, people acquire knowledge of the reality by comparing things, but not by unconditional acceptance of absolute truths, which just do not exist. So, we would like the reader to remember this note and be a delightful but critical learner of presented concepts and considerations.

IRR sometimes is called a dollar weighted rate of return (DWRR), and a money weighted rate of return (MWRR). The reasons for these naming conventions, as they are described in the literature, are vague, but this is the current terminology the reader should be aware of.

### 1.6.1. How IRR equation accounts for cash transactions inside the period

We did not formally introduce the IRR equation yet. However, equations (1.3) and (1.6) are the bases for it. There is one aspect with regard to IRR equation that we have to mention. Some people think that IRR equation does not take into account cash flows that are added or withdrawn from the portfolio during the investment period. This misunderstanding originates from the lack of knowledge about IRR equation's features and origin.

We already studied how compounding influences the accrued interest, and the total ending value. Suppose we do the interest compounding for two periods. What is the nature of interest to be added to the principal at the beginning of the second period? This is money or its equivalent. We can add more or less than the accrued interest, without sacrificing mathematical validity. We just have to understand the implications of this operation, namely that the beginning value of the next period is no longer equal to the ending value of the previous period. Once we divided the whole period into two periods, essentially, they became independent. The only thing that ties them is the interest that we brought from the first period and added to the beginning value of the second period. Figure 1.4 illustrates this consideration.

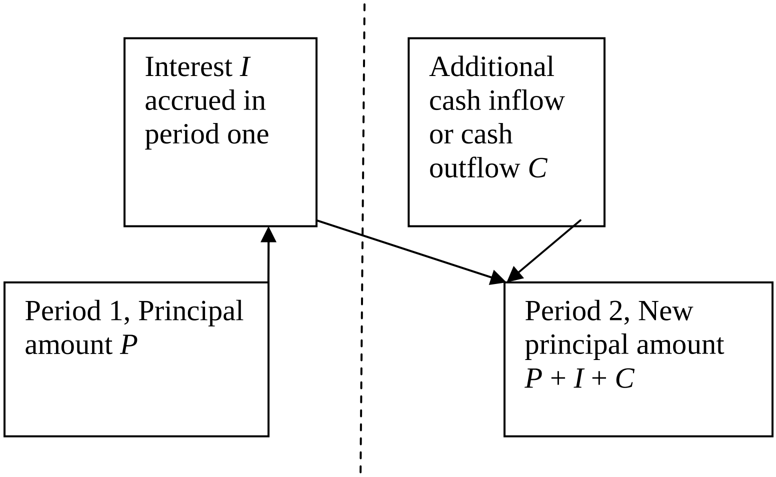



Fig. 1.4. Dividing the total period into two periods, and adding interest and cash.

Adding cash at the beginning of the second period is a legal operation both from mathematical and business perspectives. Functionally, the periods are independent. It means that the ending value of each period is completely defined by the beginning value of this period, its length, and interest rate (see formula 1.3).

What does this structuring mean to us as an argument that IRR takes into account cash flows, and does it in the only possible way? (When we work within the compounding context.) Argumentation is this. Cash flows are included into the beginning value, thus influencing the ending value for that period (equations 1.3 and 1.6). These equations define one-to-one functions. Such relation means that different beginning values produce different ending values for the same interest rate and period length. It is impossible to have different correct methods producing different numbers for the same problem, is not it?

Note that we did all derivations from scratch; we did not miss a single step without the rigorous mathematical proof. During the derivation procedure, we did not discover any points where our derivation could be split into different branches. Thus, this is the only correct derivation possible, and so, if we came to this destination, it means that there are no other alternative solutions of this problem. This proves that formulas (1.3), (1.6) and the rest of compounding related formulas take into account cash flows in the only possible way. The only difference is that in the previously considered scenarios a cash flow has been limited to a specific value, which is the interest accrued during the previous period. Next subsection presents more general case.

### 1.6.2. Deriving simple form of IRR equation

In the previous sections, we considered interest rate and applied it for clarity to money. For instance, this is the case of mortgage. However, nothing prohibits us to consider other assets whose value can be measured in money as well, such as shares, bonds, etc. In this case, we are talking about *market* value of these assets. The meaning of the beginning value and ending value slightly changes, although the value is still measured in money. The abstraction called "money" now has a little different *content*, although the *form* is the same. This process of progressive abstraction can goes on and on. At some point, such exotic things as CDOs can be added, and these market and banking explosives still can be measured in money, and even interpreted by some naive people on the same grounds as money. However, this thing is not the money anymore, the essence of this new phenomenon, the content, went a long way from the original form and, in fact, lost relationships with the origin, becoming *qualitatively* different entity. In our humble endeavors we do not go as far, and remain within the scope of traditional financial instruments, whose value is mush closely associated with the money, for instance bonds. For such instruments we will use the notion of *market* value. Accordingly, the beginning value $BV$ becomes the beginning market value $BMV$, and the ending value $EV$ becomes the ending market value $EMV$.

Mathematical proof of the additive property of cash additions to a portfolio, or withdrawals, is as follows.

$$EMV = BMV \times (1+R)^T = (P+I+C)(1+R)^T = \\ = P(1+R)^T + I(1+R)^T + C(1+R)^T \quad (1.18)$$

Thus, we proved the additive property of cash flows.

Now, let us consider the situation when cash flow is added to a portfolio at arbitrary time. Fig. 1.5 demonstrates the situation. Using notations from Fig.1.5, we can write the following formula for the total ending market value of the portfolio, exercising the additive feature (1.18) of the compounding equation.

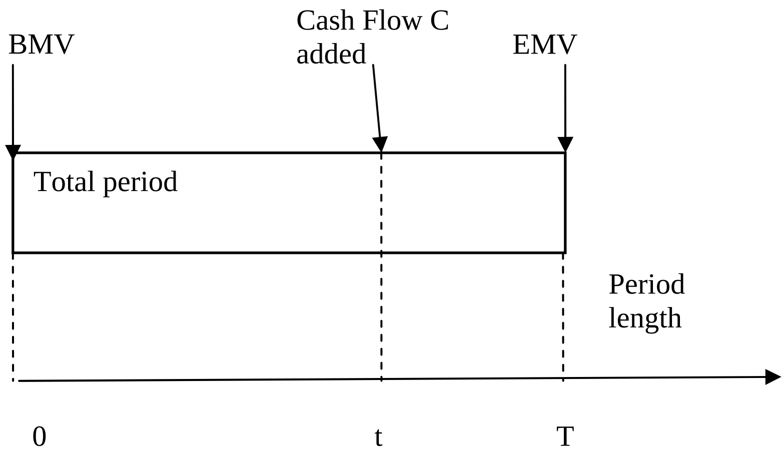



Fig. 1.5. Adding cash flow *C* at arbitrary moment *t* within a period *T*.

$$EMV = BMV \times (1+R)^T + C(1+R)^{T-t} \quad (1.19)$$

This equation is an important link in our considerations, so, we will elaborate it. The idea is that we can discount cash transaction *C* made at the moment *t* toward the beginning of the total period using equation (1.6) as follows.

$$C_b = C(1+R)^{-t} \quad (1.20)$$

where $C_b$ is some effective value of a cash flow that has to be added to the portfolio at the beginning of the total period, in order to grow to value $C$ at time $t$. The difference between values $C_b$ and $C$ is the interest that has been accrued during time $t$. This interest is equal to $C - C_b = C_b\left((1+R)^t - 1\right)$. Substitution of (1.20) into (1.18) produces equation (1.19) as follows.

$$EMV = BMV \times (1+R)^T + C_b(1+R)^T = BMV \times (1+R)^T + \\ + C \times (1+R)^{-t}(1+R)^T = BMV \times (1+R)^T + C \times (1+R)^{T-t}$$

Equation (1.19) is the IRR equation with one cash flow. Additional cash transactions can be added to the portfolio in the same way. We will discuss different forms of IRR equation in more detail in the subsequent sections.

The domain of interest rate in (1.6) is restricted by the requirement that the power base cannot be negative. Business requirement is more firm demanding also $EMV \geq 0$. (Investor cannot loose more than the beginning market value, although mathematically this business rule can be broken, and maybe life will follow the case some day.) What is important, *there are no limitations on the value of cash transactions* in the equations (1.18) and (1.19), except that it is impossible to withdraw more than the entire value of the portfolio *at the moment of the transaction.*

### 1.6.3. Non-compounding scenario

We would like to complete the discussion by showing how cash flows have to be counted in a non-compounding scenario. We can rewrite equation (1.5) as follows.

$$EMV = BMV \times (1 + T \times R) = (P + C) \times (1 + T \times R) = \\ = P \times (1 + T \times R) + C \times (1 + T \times R) \quad (1.21)$$

It means that the non-compounding scenario also has an additive property. So, if we add cash within the period, then we have, in notations of Fig. 1.3, the following:

$$EMV = B \times (1 + T \times R) + C \times (1 + (T - t) \times R) \quad (1.22)$$

This equation can be viewed of as an equivalent of IRR equation for non-compounding scenario (or non-compounding context).

We can summarize the results of this section as follows.

- Interest rate and rate of return are equivalent values with regard to their mathematical presentation. The difference is that the interest rate in our analysis is known, while rate of return has to be calculated. In fact, rate of return can be viewed as another name for unknown interest rate. Thus, the results obtained above for interest rates are applicable to situations when one considers rate of return.
- Compounding equations (1.6) and (1.19) are *the only* equations that can be derived from the definition of interest rate (rate of return).
- IRR equation is *the only correct method* to account for cash flows in a compounding scenario. This is the only formula that can be derived from the definition of interest rate to account for cash flows.
- Context of the problem, that is compounding or non-compounding, should be considered as a mandatory characteristic for investment related problems; it has to be specified unambiguously. An appropriate quantitative toolset, which is used to solve the problem, has to be consistent with the defined context.

## 1.7. Deriving a general form of IRR equation

Two scenarios will be presented in this section. First, we consider IRR equation with discrete compounding. Then, we introduce IRR equation with continuous compounding. We have to note that IRR equation has a whole set of associated terminology, sometimes ambiguous. In particular, rates of return found from the IRR equation can be called money weighted rate of return (MWRR) or dollar weighted rate of return. Please do not ask why, although some authors try to find some rationality for these names. We have never found something really convincing.

### 1.7.1. IRR Equation with discrete compounding

We will derive a general form of IRR equation with discrete compounding. In order to do that, we will rewrite equation (1.19) using the *cumulative* property (1.18) as follows.

$$EMV = BMV(1+R)^{T} + \sum_{j=1}^{j=N} C_j (1+R)^{T_j} \tag{1.23}$$

where $T_j = T - t_j$ is the time period between the moment when the cash flow occurred *till the end* of the total period, $C_j$ is the cash flow occurred at this time, and this cash flow remains in the portfolio *till the end* of the total period $T$; $N$ is the number of cash flows. All periods have to be measured in the same units of time. Cash inflow (adding cash to a portfolio) is positive; cash outflow (withdrawal from a portfolio) is negative.

The solution of this equation is the rate of return $R$ tied to the chosen *unit of time*. If $T$ is measured in months, then $R$ is the monthly rate of return, if the unit of measure is an irrational number $\pi$ (years), then $R$ is the rate of return for a period of $\pi$ years. We will refer to this unit of time as an *atomic period*.

Formula (1.23) can be transformed in different ways. We can multiply both sides of this equation by $(1+R)^{-T}$. In this case, we obtain the following equation after transformations.

$$BMV = EMV \times (1+R)^{-T} - \sum_{j=1}^{j=N} C_j (1+R)^{T_j - T} \qquad (1.24)$$

Let us denote $t_j = T - T_j$. It means that $t_j$ is a time period *from the beginning* of the total period *until* the moment when the cash flow occurs. Then, (1.24) takes the following form.

$$BMV = EMV \times (1+R)^{-T} - \sum_{j=1}^{j=N} C_j (1+R)^{t_j} \qquad (1.25)$$

This time the algebraic sign before the sum is minus. We can change it to plus by redefining cash inflows as negative values, and cash withdrawals as positive. In some literature the reader can find exactly this approach. However, this book, as well as many others, treats cash inflows as positive values, which make sense with regard to portfolio value.

IRR equation in the form (1.25) discounts the ending market value and cash flows to the beginning of the total period, which is the present value. Both forms of IRR equation are used in business practice. The choice depends on the business case and personal preferences.

There is one remarkable and practical thing about equation (1.23). The beginning market value is not something special in this formula. It is not distinguished from other cash transactions, and can be viewed as another cash flow done at the beginning of a total period. This consideration brings more uniformity to the approach, and simplifies an implementation of computing algorithms and system design. Equation (1.26) implements these considerations in the following mathematical form.

$$EMV = \sum_{j=0}^{j=N} C_j (1+R)^{T_j} \qquad (1.26)$$

where $C_0 = BMV$; $T_0 = T$.

We will use both this and traditional forms of IRR equation. However, it is important to understand that the beginning market value is nothing else but the cash flow originated at the beginning of investment period.

### 1.7.2. IRR equation with continuous compounding

IRR equation with continuous compounding is based on formula (1.17). The only difference between the IRR equation for the discrete compounding and continuous compounding is that cash flows (including the beginning value) are compounded continuously. It allows us to rewrite IRR equation (1.23) for the discrete compounding as follows.

$$EMV = BMV \times e^{RT} + \sum_{j=1}^{j=N} C_j e^{RT_j} \qquad (1.27)$$

where notations are the same as in formula (1.23), that is the value of $T_j = T - t_j$ is the time period when the cash flow $C_j$ occurred, *till the end* of the total period $T$; $N$ is the number of cash transactions.

As previously, all periods have to be measured in the same units of time. Cash inflow (adding cash to a portfolio) is positive; cash outflow (withdrawal from a portfolio) is negative.

Similarly to equation (1.24), formula (1.27) can be rewritten to discount cash flows to the beginning of a total period. After the appropriate transformations, we obtain the following equation.

$$BMV = EMV \times e^{-RT} - \sum_{j=1}^{j=N} C_j e^{-t_j R} \qquad (1.28)$$

where $t_j$ is a time period *from the beginning* of the total period until the moment when the cash flow occurs.

IRR equation (1.28), similarly to equation (1.25), discounts the ending market value and cash flows to the beginning of the total period (to a present value). As in case with the discrete compounding, the choice depends on the business case, type of computations and personal preferences.

We can write a more compact form of the IRR equation for continuous compounding similarly to equation (1.26), considering the beginning market value *BMV* as a cash flow that originated at the beginning of the total period.

$$EMV = \sum_{j=0}^{j=N} C_j e^{RT_j} \tag{1.29}$$

where $C_0 = BMV$; $T_0 = T$.

### 1.7.3. Generalization of continuous compounding. Force of interest

Before, the discussion in this section covered the main forms of IRR equations that describe continuously compounded cash flows in the investment portfolio, while assuming that interest rates (or rates of return) are constants. However, this assumption may not be always valid. With this regard, there is a more complex notion associated with continuous compounding, which is called *force of interest*. The description of this concept can be found in textbooks such as (Broverman, 1991). Note that it may have slightly different interpretations in the literature.

Until now, we considered interest rate and rate of return as constants related to a certain period of time. However, these values can depend on time, and maybe some other parameters. We can represent this dependence as $R = R(t, M)$, where $M$ denotes a set of parameters influencing the interest rate beside the time. Examples of these parameters are seasonal temperature, hurricane feasibility, or the intensity of the credit market crunch, media coverage of presidential debates, etc. In this case, equation (1.29) takes the following form.

$$EMV = BMV \times \sum_{j=0}^{j=N} C_j e^{\int_{T_j} R(t,M)dt}$$

where integration is done over the $T_j$ period. We consider $M$ as a constant vector, although in a more general case this set of parameters can depend on time as well.

Dependence of interest rate on time and other parameters can be embedded in a similar way into the IRR equation for discrete compounding. However, we will not discuss this topic in detail. The goal of this book is to explore comprehensively existing *practical* approaches in detail, starting from the very basic assumptions these methods originated from, and propose and describe remedies and recommendations for some inconsistencies and drawbacks which the present methods have. It is a normal way of development for any discipline, and financial mathematical applications are not exceptions. As we see the situation, a lot to be done in this area. Researching of more complicated scenarios, such as the aforementioned force of interest, is certainly an interesting problem, and the efforts in this direction should be appreciated. Maybe at some point we will embark into this adventure too. However, there is such a thing called *priority*, and from practical perspective the state of existing methods, which are used today in everyday business practice, is more demanding. This is why we do not consider less common applications.

As a side note, we have to say that although mathematical expressions for the forced interest may look cumbersome, its mathematics is not very complicated, and many readers probably can do all the necessary transformations on their own, when there is a need. Finding solution for the force of interest equation is possible, as soon as the dependence of interest rate on time is known. If there is no analytical expression for the integral, then numerical methods can provide a solution.

What is more demanding, is finding a solution for this equation with regard to function $R(t,M)$. Solving integral equations in general is a challenging task. Solutions of such equations exist when some specific conditions are satisfied, most of which relate to the kernel function of an integral equation. Fredgholm's theorems allow investigating the solution's feasibility, but this is where things are becoming mathematically non-trivial. So, some tradeoff has to be found between the complexity of this approach and the expected benefits.

### 1.7.4. Getting the "look and feel" of IRR equation

As in every area of human activity, it is important to develop a "feeling" for the IRR equation behavior. It means the understanding of relationships between its parameters, the boundaries of domain and range, and intuitive comprehension of this *entire* phenomenon. We should develop the feeling of the IRR's asymptotic behavior, extreme points and roots, convexity and concavity, as well as dynamics of its parameters and their co-influence.

Let us rewrite formula (1.23) as a function of portfolio's parameters, in order to facilitate this task.

$$F(R) = BMV(1+R)^T + \sum_{j=1}^{j=N} C_j (1+R)^{T_j} \tag{1.30}$$

where $F(R)$ is a function of argument $R$.

We will call $F(R)$ as IRR function. The intersection of a linear function $L(R) = EMV$ and $F(R)$ defines the solution of equation (1.23). The domain of function $F(R)$ is all real numbers $R \geq -1$. Function $F(R)$ asymptotically approaches the function $BMV = (1+R)^T$, when $R \to \infty$. This function grows monotonically when $R$ increases. It can be shown by rewriting equation (1.30) as follows.

$$F(R) = BMV(1+R)^T \left[ 1 + \sum_{j=1}^{j=N} \frac{C_j}{BMV(1+R)^{T-T_j}} \right] \tag{1.31}$$

Then we have

$$\lim_{R\to\infty} F(R) = \lim_{R\to\infty} BMV(1+R)^T \left[ 1 + \sum_{j=1}^{j=N} \frac{C_j}{BMV(1+R)^{T-T_j}} \right] =$$
$$= BMV \times (1+R)^T$$

(1.32)

We have taken into account in formula (1.32) that $T > Tj$. The beginning market value is always non-negative. Accordingly, the right side of (1.32) is always non-negative value. If $BMV > 0$, and $R > -1$, then this right side is always positively defined. It means that the number of roots in equation (1.23) is always finite.

Fig. 1.6 demonstrates this feature of IRR equation in graphical form. Intersection of the IRR function and the line $L(R) = EMV$ defines the root (or roots in general) of equation (1.23), which is the value of $R_0$ in Fig. 1.6. The following data have been used.

Table 1.1. Sample portfolio used for graphs in Fig. 1.6.

| | | | | | |
|---|---|---|---|---|---|
| Cash Flows $C_j$ | 1.5 | 1.0 | -1.0 | 1.0 | -4.1 |
| Periods $T_j$ | 1.0 | 0.9 | 0.6 | 0.2 | 0.1 |

Of most interest in the IRR function is the part near the solution of equation (1.23). The IRR equation is often blamed in ambiguity of its solution for the same data. This is a quite natural state of affairs given the polynomial nature of the IRR equation. In general, the number of unique roots for a polynomial equation can be found using Sturm's theorem (Sturm's theorem, in reference). A maximum number of *possible* positive and negative roots can be estimated using Descartes' Sign Rule (Descartes', in reference).

In particular, if all cash flows are positive, then every term in $F(R)$ is positive, and there are no changes in algebraic signs of the polynomial coefficients at all. All terms are monotonically growing functions. So, the resulting function $F(R)$ is also a monotonically growing function. Consequently, it can have only one point of intersection with the linear function $L(R) = EMV$, and hence the only solution (we assume $EMV \geq 0$, which is a reasonable assumption).

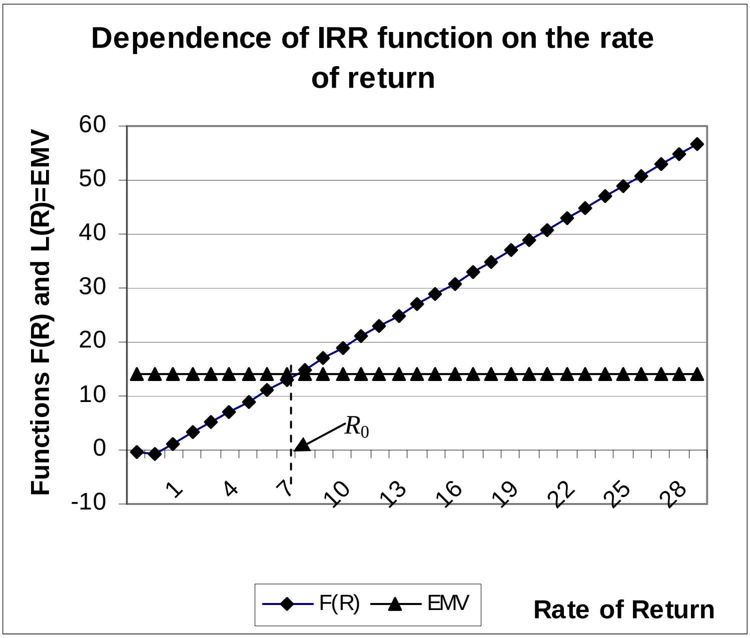



Fig. 1.6. Dependence of IRR function on the rate of return. Horizontal line is a linear function *L*(*R*) = *EMV*.


Descartes' rule is applied to integer powers. However, it can be transformed to deal with the rational powers as well. We just have to find the common multiplier for the powers. Then (suppose this is the number *m*) we denote $y=(1+R)^{\frac{1}{m}}$ and rewrite the original expression in integer powers. This substitution does not add any additional roots, because $R \geq -1$. It means that the substitution preserves equivalence. If the power is an irrational number, then we can approximate it by a rational number as close as we want. We have already mentioned that equation (1.23) has a finite number of roots. So, we will not loose roots of the original equation by choosing the appropriate approximation, although we can acquire additional solutions.

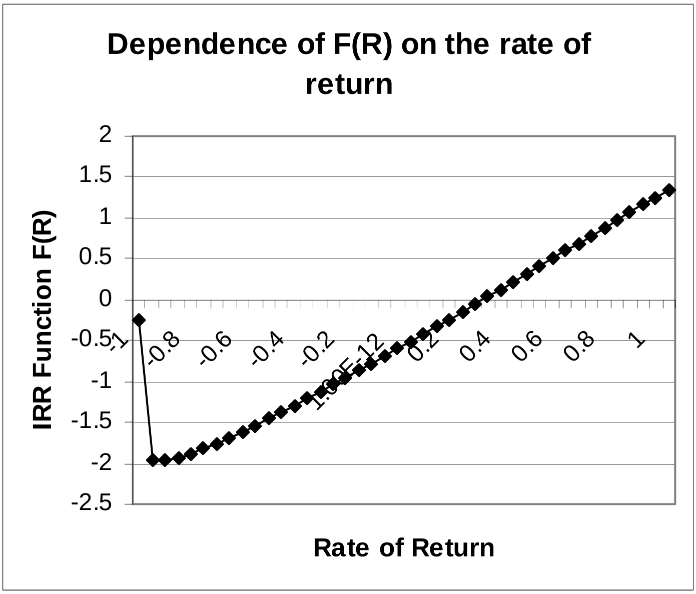



Fig. 1.7. Dependence of IRR function on the rate of return in a smaller domain.

Fig. 1.7 enlarges the same graph of function $F(R)$ from Fig. 1.6 in a smaller domain of rate of return. It has one minimum, after which it increases monotonically, and approaches the asymptotic curve which we discussed before. The ending market value cannot be less than zero. It means that equation (1.23) in this case has one unique solution, because there can only be one intersection of $F(R)$ and $L(R)$ in the range $F(R) \geq 0$ .

These are preliminary notes with regard to controversial problem of unknown number of solutions for the IRR equation.