A HIERARCHY OF METHODS FOR CALCULATING RATES OF RETURN

This article explores two topics. The first is the relationships between mathematical algorithms for calculating rates of return on investment portfolios. Article introduces and describes a hierarchy between different methods. It is shown that mathematically the Internal Rate of Return (IRR) is the most adequate method among the all presently used approaches. It should be used as a standard or a true value all other methods have to refer to. Secondly the article describes new mathematical algorithms that overcome many drawbacks of the present approaches. These new methods are collectively called Shestopaloff's linking (SL) methods. The following advantageous aspects of SL methods are considered – analytical research, system performance and system design. Analysis demonstrates the benefits SL methods provide in all of these areas. Detailed consideration is given to how new methods can be introduced into everyday business practice.

Yuri Shestopaloff, Ph.D.
is the director of technology for SegmentSoft Inc.. He is a Full Professor, Doctor of Sciences and Ph.D. in mathematical methods for data interpretation with a master's degree in Electrical Engineering. His areas of interest are developing mathematical algorithms for data interpretation and data processing in the financial industry, designing and developing middle and large-scale distributed software applications and systems, including performance measurement and trading systems. Yuri has been published in over eighty professional publications. He also published a series of articles on the philosophy of science.

Alex Shestopaloff
is a Mathematician at SegmentSoft, Inc.. Alex had worked previously as a senior system administrator with responsibility for the system design, implementation and deployment of high performance distributed server systems including internet-based ones. He has won awards for system design and also has received an Honorable Mention Award for his article about
Consistent linking in The Journal of Performance Measurement *in 2006. He achieved his degree from the University of Toronto in Statistics.*

Available Mathematical Methods For Calculating Rate of Return

There are different mathematical methods for computing rates of return on an investment. Detailed descriptions of these can be found in (Spaulding 1997), (Spaulding 2003), (Feibel 2003), and (Chestopalov, *et. al.*, 2005). In this article, we consider four main mathematical approaches.

First we will consider IRR (internal rate of return). It is also widely known as money weighted rate of return (MWRR) and dollar-weighted rate of return. Presently, its significance is growing. This trend has been officially recognized at the Performance Management, Attribution and Risk Conference in Philadelphia (PMAR 2006) and appropriate proposals have been submitted. For example, GIPS standards explicitly state the application of IRR for the private equity as follows: “Private Equity Calculation Methodology - Requirements. Firms must calculate the annualized since-inception internal rate of return (SI-IRR).”

There were important findings related to the more objective nature of MWRR for many performance management applications. There is also a clear trend among clients toward MWRR because of its objectiveness and clients’ growing understanding of that fact. As an example, it is very beneficial from the perspective of decomposing rate of return for the analytical and performance management purposes. Details and a comprehensive bibliography can be found in (Illmer 2003).

The next method is time-weighted rate of return (TWRR). Sometimes it is called the “true time-weighted rate of return,” in order to distinguish it from the time-weighted rate of return calculated by the Modified Dietz formulae.

It should be noted, however, that presently GIPS refers to TWRR as a “*calculation that computes period-by-period returns on an investment and removes the effects of external cash flows, which are generally client-driven, and best reflects the firm's ability to manage assets according to a specified strategy or objective.*” We will show that the TWRR fails to meet these objectives.

The next approach is known as the Modified Dietz method. However, the status of the Modified Dietz formula is not finalized because of the different interpretations. For example, in (Feibel 2003) one can find on pg. 43 the following: “ . . .rates computed with either the exact IRR or the Modified Dietz method are money-weighted returns.” At the same time, in (Spaulding 1997) and other publications, Modified Dietz formulae are considered to approximate the time-weighted rate of return. The last approach is the most adequate one to the nature of the equation. The rationale behind the latter view is that the Modified Dietz formulae are considered to give a reasonable approximation to the time-weighted rate of return whenever the cash flows are small. In any case the Modified Dietz formula is an important instrument in performance measurement business.

Geometric linking is different from the previous methods. It is not a self-sufficient method. It is used in conjunction with other methods as a vehicle which allows linking of returns for *sequential* periods, given the rate of return for each included period calculated by methods for a single period return.

Why do so many methods exist for calculating the rate of return and why do they generally produce different results? One widely accepted explanation is that this is due to the diversity of business scenarios and business requirements, which particular methods address. For example, it is claimed that MWRR and Modified Dietz are good for the longer periods, while TWRR is suitable for the shorter ones. However, this was never demonstrated convincingly via rigorous quantitative or analytical studies. In fact, for the example given above, analysis of mathematical expressions for MWRR and Modified Dietz formulae does not reveal any advantages or limitations related to period length, as shown in (A. Chestopalov 2005).

The major source of the industry's ambiguities is the absence of a reference point to which all methods should be compared in order to find its validity and the scope of applicability. If there were a clear understanding what method is mathematically correct, then many problems would not appear at all. It is shown below that from a mathematical prospective this method is IRR (MWRR). The problem using it until recently was a technical one – laborious computation. With the computing resources available today, this is not a problem anymore. There are also mathematical advances that make the finding of a numerical solution of the IRR equation substantially easier (I. Chestopalov 2004), (A. Chestopalov 2005).

Another consideration of a general nature in favor of paying closer attention to this problem is as follows. The way towards objectivity in this industry is unification, not the diversification of methods. The rationale for that is that the problem of calculating the rate of return is determining one number in some universal way independent of the innumerable business scenarios. Otherwise the subsequent development of the whole industry will bring confusion, because the rate of return in most instances is the first input in many other applications, such as attribution, risks adjusted rate of return and many others. Within the current situation it means that the number of possible financial characteristics derived or based on rates of return will be increasing exponentially and eventually will produce enormous, if not unmanageable, number of relationships, and specific cases.

Money-weighted Rate of Return

We will show below that the money-weighted rate of return (MWRR) is the main method from which other methods can be derived. If one is looking for the most objective parameter characterizing the return on an investment portfolio, this is the MWRR. If one wants to develop the most adequate model of this phenomenon, then MWRR has to be included in order to provide the best possible adequacy of the model with the reality. MWRR method is used presently in the industry. However, it is considered *as one of the* methods. It is treated as an independent standalone method useful for some purposes and not good enough for the others. The same situation is with other methods. Each method is considered as an independent entity, equal among equals. The last approach is good for the political reverences and campaigns but when it comes to reality it turns out that every phenomenon has some structure and inner dependencies. The parameters defining the phenomenon are interconnected not in an arbitrary but in a certain way. Otherwise the phenomenon would not exist because it would be shapeless.

One of the main myths within the investment performance industry is this: there are many rates of return, each servicing some particular specific need. For example, the investor is interested in a total return. The fund manager's performance has to be evaluated differently, because he has no control over the cash flows and transactions initiated by the investor. The so-called time-weighted rate of return TWRR (though it has nothing to do with the time-weighting) is supposed to fulfill this prophecy. In reality, MWRR already takes into account all cash flows and transactions. However, the myth is so widespread and canonized within the industry that even a hint of touching this sacred pillar produces a rage combined with allegations of heresy. The TWRR is really naked, but the powerful inertia of the public warship seems to blind everybody. So, we would like the reader to remember this situation and be a critical but fair judge of our arguments and concepts.

The specifics of the MWRR are that there is no generally analytical solution for the non-linear equation used for finding this rate of return, except in some particular cases. It can be solved only numerically. However, the MWRR is the only true rate of return that mathematicians accept as correct because only the MWRR can be derived mathematically from the definition of the rate of return. It should be understood that mathematical correctness does not imply the numerically precise value, as some people think. Most of the existing mathematical methods principally produce only approximate numerical values. However, it does not impede their usefulness in real applications, because it is possible to find numerical values with a required accuracy. Otherwise neither the technology nor the science progress would be possible at all if humankind relies only on the precise numerical values. The same is applied toward the IRR – MWRR method. It is a *correct* and *precise* method from the mathematical prospective, but its numerical solution is an approximate value that always can be found with a predefined accuracy.

All other methods are mathematically *approximate* methods for calculating the rate of return. They may produce in many instances the precise numerical result for the data substituted into the appropriate formula. However, the formula itself principally is an approximate one. It is not unlike finding the area of a circle. There is only one true value for the area: πr^2, where r is the circle's radius. However, the area cannot be calculated exactly because we always work with an approximation of the number π. We can also approximate the circle's area by an inscribed or circumscribed hexagon or octagon. This approximation will work for some purposes, but nobody will insist that this is a legitimate replacement of the circle's area on all occasions. There is only one true number for the area of the circle. The situation is similar with a rate of return. The Modified Dietz method and TWRR both are just mathematical approximations. The reality, however, is that TWRR is often considered to be an equally important alternative to MWRR, while it is about as good an estimate of the mathematically correct rate of return as the area of an inscribed octagon is an estimate of a circle's area.

Deriving IRR – MWRR equation

The MWRR equation is *the only* equation that can be derived from the definition of rate of return. There are no branches during the derivation process that can lead to a different result. The definition of rate of return is as follows:

$$R = (EMV – BMV) / BMV \quad (1)$$

where EMV – ending market value, BMV – beginning market value, R – rate of return.

Solving (1) we find the ending market value.

$$EMV = BMV + BMV×R = BMV×(1 + R) \quad (2)$$

Please note that the same definition is valid for the interest rate, so that these terms can be considered as synonyms in the discussed context. Suppose the lender lent the money for two periods (let's say for two years) and expects interest payments at the end of the second period. However, the rate of return agreed upon is one period. Interest accumulated during the first period will be added to the principal. During the second period, interest will be calculated on this total amount. The ending value of the first period will be the beginning value of the second period. Substituting formula (1) we have:

$$EMV_2 = BMV (1 + R) = EMV_1 (1 + R) = BMV_1(1 + R)(1 + R) = BMV (1 + R)^2 \quad (3)$$

where index '2' relates to the second period.

Similarly, we can add the third period and so on, so that eventually we will begin to suspect that for the n^{th} period the ending value will be

$$EMV_n = BMV×(1 + R)^n \quad (4)$$

However, our wild guess is not a mathematical proof yet. Strict mathematical proof can be done using a method of mathematical induction. So, formula (4) is valid according to the principle of mathematical induction for any number $n \geq 0$. We can do a very similar exercise when the power is negative.

We should revisit formula (4) that was derived with the assumption of an integer non-negative power. Generalization of formula (4) for the real power is a consequence of additive property of powers when the base is the same. That is:

$$B^d = B^{a+b+c} = B^a B^b B^c \text{ , if } d = a+b+c.$$

There are other ways to do the same generalization. So, formula (4) can be rewritten as follows

$$EMV_T = BMV×(1 + R)^T \quad (5)$$

where T is a real number denoting the period's length in units of time to which the rate of return is applied. Please note that the domain of applicability for the formula (5) is restricted only by the requirement that the power base cannot be negative, so that $R \geq -1$.

The Inherent Relationship of Compounding and Cash Flows

We did not yet formally introduce the MWRR equation. However, equations (4) and (5) are the base of it. So, it is reasonable to describe the other aspects of the general attitude toward the MWRR equation at this point. There is a wrong belief in the industry that the MWRR equation does not take into account the influence of cash flows onto the rate of return properly. This misunderstanding originates from the lack of knowledge and coherent understanding of the nature of the MWRR equation. Let's look at how compounding influences the interest accrued and the total ending market value. Suppose we do compounding for two periods. What is the nature of the interest to be added to the principal at the beginning of the second period? This is just money. We can add more or less than the interest accrued. Once we divided the whole period into two, they became essentially independent. The only thing that ties them is the interest we brought from the first period and added to the beginning market value of the second period.

However, functionally, the periods are independent as long as we do correct calculations within the period based on equation (5). It means that the beginning market value of the second period can be any one and our calculations still are absolutely legal. The ending market value will be calculated in a right way because there are no any other factors influencing the result of calculation except the beginning market value, interest rate and the length of the period (see Equation Five). What does this structuring mean to us in the argument that MWRR takes into account the influence of cash flows onto the rate of return properly and does it in the only possible right way? Any fluctuations in the cash flow will be reflected in the ending market value because they constitute the beginning market value of the period that influences the value of the ending market value (Equations 4, 5). These equations define unambiguous one-to-one functions, so that different BMVs produce different EMVs for the fixed interest rate and the period length. (We cannot have different correct methods producing different numbers for the same situation, can we?)

Why MWRR is the only correct method to account for the cash flows properly? The reader remembers that we did all the derivation from scratch. We did not miss a single step without proving it mathematically. During the procedure we did not discover any points where our derivation could be split into different branches. It is the only correct derivation possible. So, if we came to this point it means that there are no other points at all we could arrive at. That proves that Equations 4 and 5, and the rest of compounding related formulas inherently take into account cash flows, though in this particular scenario restricting it to the specific value – to the interest accrued during the previous period. Summarizing what was said above, we can write a very important expression proving the additive property of interest accrued on any amount added to the portfolio.

$$EMV = BMV\times(1+R)^T = (P+I+C)\times(1+R)^T = P\times(1+R)^T + I\times(1+R)^T + C\times(1+R)^T, \quad (6)$$

where P is the principal amount; I – interest accrued during the previous period; C – cash flow.

What if we decide to add cash flow somewhere down the road at any arbitrary moment? We can write the following expression for a total ending market value of the portfolio exercising the additive feature (Equation 6) of the compounding equation

$$EMV = BMV\times(1 + R)^{T} + C\times(1 + R)^{T-t} \quad . \qquad (7)$$

Here T is the total period, t – is the time from the beginning of the period when cash flow has been added to the portfolio.

Equation (7) can be proved as follows using additive properties expressed by equation (6). We can discount cash transaction C made at the moment *t* toward the beginning of the total period using equation (5) as follows.

$$C_b = C\times(1 + R)^{-t} \qquad (8)$$

where C_b is some effective value of cash flow that has to be added to the portfolio at the beginning of the total period in order to grow to value C at time t. The difference between values C_b and C is the interest accrued during time *t* equal to $C_b\times(1 + R)^{t.}$ Substituting (8) into (5) produces the equation (7) as follows:

$$EMV = BMV\times(1 + R)^{T} + C_b\times(1 + R)^{T} = BMV\times(1 + R)^{T} + C\times(1 + R)^{-t}\times(1 + R)^{T} = BMV\times(1 + R)^{T} + C\times(1 + R)^{T-t}$$

This completes the proof of validity of Equation (7). It is IRR - MWRR equation with one cash flow. In the same way additional cash transactions can be added to the portfolio and taken into account in order to produce the general form of IRR – MWRR equation.

The domain of applicability for Equation 5 is restricted by the requirement that the power base cannot be negative. Business requirements are more rigid. They demand also that the ending market value cannot be negative, that is $EMV \geq 0$. (One cannot lose more than the beginning market value, though mathematically this rule can be broken.) However, what is important, *there are no limitations on the value of cash transactions* in Equations 6 and 7, except that it is impossible to withdraw more than the whole portfolio's value.

IRR - MWRR equation (7) takes into account cash flows in the most comprehensive and natural way possible, so arguments that claim that it does not do this have no grounds.

We will not discuss different forms of MWRR equation that naturally follows from Equation 7, as well as the case of multiple cash flows because the appropriate formulas are direct derivatives from Equation Seven. Our purpose was to show how cash flows are accounted for in the MWRR equation if it is derived from the very definition of rate of return. The second goal we achieved is the proof that there are no other formulas taking cash flows into account that can be derived from the definition of the rate of return.

Other Methods as Direct Derivatives of MWRR

One of the additional analytical proofs that MWRR is the primary method can relate to the fact that the Modified Dietz formula is a direct derivative from the MWRR equation as its approximation by Taylor series (I. Chestopalov 2004). Modified Dietz formula in its turn relates to other methods. For example, sub-period returns used in TWRR can be easily derived from Modified Dietz formula as a particular case. Modified Dietz formula is as follows

$$R_0 = \frac{EMV - BMV - \sum C_j}{BMV + \sum C_j T_j} \qquad (9)$$

where C_j denotes cash flows, T_j is the appropriate period length.

Assuming that cash flows occur at the beginning of periods, one obtains from the Dietz formula:

R = (EMV-BMV-C)/(BMV+C) = EMV/(BMV+C)-1,

The last expression is a sub-period return used in TWRR.

It was shown (A. Chestopalov, 2005) that geometric linking is a special case of the Shestopaloff's linking method when there are no cash transactions. (Previously Shestopaloff's linking methods were introduced as consistent linking methods.) Equation (10) below from the cited work demonstrates this point convincingly. Only the first term is left in the absence of cash flows that is nothing else as the definition of the geometric linking procedure.

$$E_N = B_1 \prod_{n-1}^{N} (1+R_1)^{Tn} + \sum_{n=1}^{N-1} \sum_{i=1}^{I_n} C_{ni} (1+R_n)^{T_{ni}} \prod_{k=n+1}^{N} (1+R_k)^{T_n} + \sum_{i=1}^{I_N} C_{Ni} (1+R_N)^{T_{Ni}} \qquad (10)$$

Shestopaloff's linking for the Modified Dietz formulae produces geometric linking as a particular case. Given the fact that Shestopaloff's linking uses MWRR (IRR) equation as a starting point, it means that geometric linking is also a special case of MWRR when MWRR applies to multiple periods.

At the same time there are no reverse mathematical relationships between MWRR and any of the other existing methods. Such "reverse engineering" is impossible because all other methods implement *particular* restrictive cases of MWRR equation, thus *loosing* *generality*, while MWRR implements the most general case applicable to any scenario without restrictions on the value of cash transactions relative to the total portfolio, frequency, period lengths, etc. It has to be mentioned that current standard TWRR explicitly restricts the area of its applicability, up to the point of specifying the maximum percentage of allowable cash transactions to preserve some reasonable but still unknown accuracy. Knowing valuation error is a very important issue, but TWRR principally does not have an answer for that. The only way to do this is to compare TWRR with the true value of rate of return (which is, as we proved already, MWRR), but at the moment the industry does not have an official one. TWRR in itself is an official industry standard. Another limitation imposed on the usage of TWRR is that it forces period fragmentation to be done at the time of cash transactions that often leads to awkward situations in the analytical studies.

Let's consider simple example how TWRR can produce obviously inadequate results. Suppose fund manager manages the assets in the following manner represented in the table below. That is, he lost money in the first period, deducted his commissions at the beginning of the second period and was lucky enough in casino putting all the remaining money at stake instead of resorting to the infamous "O'Hara spread" (O'Hara is an Airport in Chicago).

	Beginning market value	Cash transactions	Ending market value
Period 1	1,000,000		10,000
Period 2	10,000	9,900	10
Period 3	10		20000

Then the manager's performance evaluated with the aid of TWRR is

$$R = (-0.99 + 1) \times(-0.9 + 1)\times(1999 + 1) – 1 = 1, \text{ that is } 100\ \%. \quad (11)$$

Common sense hints that this manager should not be praised as a high achiever because he lost investors' money actually. Nonetheless the number evaluating his performance is great. The problem is not even the number because its nonsense is obvious. The problem is that there is no way to figure out the applicability domain of TWRR method because of its heuristic ungrounded nature.

The base of TWRR is geometric linking. However, geometric linking is valid *only* when the beginning market value of the next period is equal exactly to the ending market value of the previous period. This is *the only* domain of its applicability. Any attempt to apply the geometric linking operation outside its domain inevitably leads to problems.

So, we proved mathematically that MWRR is the most general method for calculating the rate of return, and all other methods are its particular cases, but not vice versa. Secondly, MWRR is derived on the basis of very general assumptions. Other methods are derived based on more restrictive initial assumptions and sometimes just on heuristic approaches like Modified Dietz method or TWRR. This explains the fact that the realm of applicability of MWRR is wider than area of applicability of any other method and includes areas of application of all other methods. MWRR is universally applicable to any possible scenario.

These considerations about generality of MWRR and the restrictiveness of other methods may sound like a subtle issue, but from the view of scientific verification methodology this is an important and valuable consideration. The reason for the appearance of the approximate methods was complexity of solving the IRR equation in the days when computers were just on the horizon, as well as lack of mathematical education among the users. If the performance measurement business had to begin today and the people were mathematically literate enough, then MWRR would be the only rate of return in use.

Different numerical methods can be used for solving MWRR equation. However, all of them benefit if the first approximation is close to a true solution. For that purpose in (I. Chestopalov 2004) two methods for approximate finding MWRR were proposed and in (A. Chestopalov 2005) were developed further. One uses linear approximation of MWRR equation by Taylor series, the other is a quadratic one. Application of these formulae towards the real portfolios demonstrated high accuracy of finding the MWRR value. If more accuracy is required, then these solutions can be used as a first approximation for numerical methods, thus significantly enhancing application performance.

Comparison of Methods

Let us make a note about geometric linking. The problem with geometric linking is that it produces incorrect result if cash flows and transactions occur within the same period (or on the boundary) – which is the way the investment industry operates. Geometric linking is a direct consequence of formula five. However, this formula has been derived with the assumption that the ending market value of the previous period is the beginning market value of the next period (formulas three and four). We noted already that this is the only applicability domain of the geometric linking. Stepping outside the applicability domain of some formula is like bringing summer clothes to Antarctica. One might survive for some time on the fringes of the continent in a summer time, but the rest of his fate is murky. TWRR shares this adventure one-to-one. The reason is that TWRR is based on geometric linking approach while it was introduced outside the applicability domain of geometric linking.

The correct way of introducing some new approach is to research the domain of applicability of the foundation first. Then, if it is not sufficient, the appropriate generalization to be done to cover the new area of application. A good example of such consistent and correct approach is the derivation of equation seven.

The following example with a hypothetical portfolio includes rates of return calculated using different methods. The whole period is composed of two equal periods. At the end of the first period portfolio's market value has been increased to $2,000,000. Assets worth $1,999,900 were sold and cash has been withdrawn from the portfolio, so that at the beginning of the second period the portfolio's market value is $100.

	Beginning market value, $	Cash transactions, $	Ending market value, $
Period 1	1,000,000	-1,999,900	100
Period 2	100	0	500

Calculating rate of return according to all considered methods produces the following results.

	Period 1	Period 2	Total return
TWRR	1.0 (100.0%)	4.0 (400%)	9.0
Modified Dietz			20,008
MWRR (IRR)			3.0006

It is worth to note how volatile both TWRR and Modified Dietz methods in this scenario are. Small market values in the second period do not influence much the overall rate of return calculated by MWRR. This is reasonable because the bulk of asset's growth is associated with the first period. However, minor changes of these values within the second period influence drastically the value of return calculated using TWRR method. For example, if the ending market value of the second period is $5000, then TWRR = 99, while MWRR is not changing much in this case. If, for example, we reduce the length of the second period to one tenth of the overall period, then Modified Diet method produces 1.25 instead of 20,008. This high volatility of methods is another evidence of its restricted range of applicability compared to MWRR.

Whatever input values are, whatever extreme scenarios are considered, MWRR always produces correct, true rate of return. It is impossible to find the situation in which MWRR does not produce correct rate of return from all perspectives, however extreme the situation is. At the same time, it is possible to create portfolio for any other method, such that when the method is applied, the results will be in disagreement with common sense as well as with other methods. One can do similar calculations for the extreme portfolio shown below. MWRR will be the only method producing reasonable result.

	Beginning market value	Cash transactions	Ending market value
Period 1	1,000,000	-1,999,999	1
Period 2	1	0	500

Wide scope of applicability of MWRR and the corresponding limited applicability of other methods is another convincing proof that MWRR is the primary method for the calculation rate of return.

Figure 1 illustrates the situation. Let F(R)=0 be the MWRR equation and MW be the true MWRR. Then all methods mentioned earlier produce solutions with different accuracy compared to MWRR.

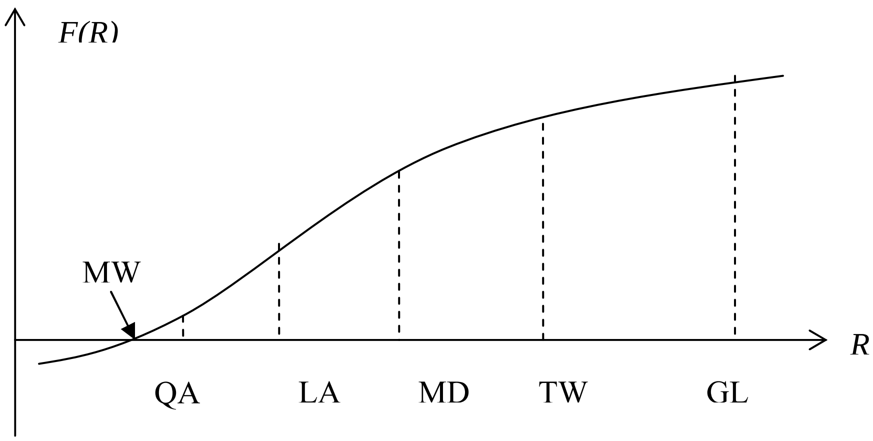

Figure 1. Relationship of Rate of Return Calculated by Different Methods.
MW – money weighted rate of return, TW – time weighted rate of return, GL – geometric linking, LA – using linear approximation of MWRR equation, QA – using quadratic approximation of MWRR equation, MD – Modified Dietz formula.

The relative accuracy of each of the mentioned approaches compared to MWRR can be different, depending on the particular scenario. It may happen that in certain situations MD will be closer than TW, and vice versa in other situations. This is another valuable feature of MWRR. It can be used as a consistent reference point to judge the accuracy of any other method.

It is often heard that the rates of return are approximate by nature. The prices are approximate ones, the period's starting point can be set differently and so on. Such consideration often serves as justification of using mathematically incorrect approaches. However, when one gives approximate estimation he should provide the error estimation. No error estimation can be done in the case with TWRR or Modified Dietz. So, the investor is essentially is left in the darkness how truthful the reported results are. The issue growth when it comes to analytical studies exercising chain calculations. It is well known that systematic errors accumulate quickly in such situations. There is no satisfactory answer to this question as well. One can find recommendations kind of those that cash transactions should not exceed a certain percentage of the total portfolio. However, in reality these recommendations come from nowhere, just a gut feeling.

<u>Summary of MWRR Features</u>

1) MWRR is the only method for calculating rate of return that can be derived from the definition of rate of return.
2) TWRR and Modified Dietz methods for computing the rate of return are derivatives of MWRR. They represent special restricted cases, approximations of MWRR.

3) MWRR always produces a reasonable result, the true rate of return for *any* possible business scenario, while all other methods have narrower, limited scope of applicability beyond which they produce incorrect results.
4) MWRR is a superior method above all other approaches for calculating rate of return in terms of generality, consistency and universal applicability. Thus it is naturally a perfect standard for the rate of return to which all other methods have to be referred.

Figure 2 summarizes the hierarchy of methods for calculating the rate of return in the graphical form. We did not consider other known methods. However, they can also be added into this scheme. None of them can challenge the supremacy of MWRR. Their origin can be traced to MWRR in one way or another.

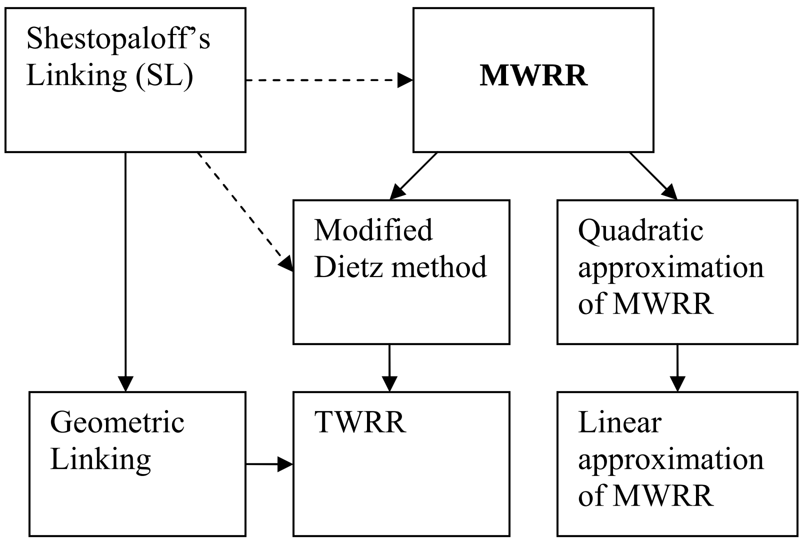

Figure 2. Hierarchy of Mathematical Methods for Calculating Rates of Return.

In Figure 2, quadratic and linear approximations are the ones introduced in (A. Chestopalov 2005). A solid line is directed from the parent method, dashed lines denote the area of applicability. Note: Shestopaloff's linking can be also applied toward the linear and quadratic approximations of MWRR.

<u>Limitations of Existing Methods when Applied to the Investment Analysis</u>

The principal problem with the existing methods is a contradiction between the nature of investment business, a highly dynamic environment with uncertain outcomes, on one side, and fairly rigid and numerous mathematical methods currently servicing this industry, on the other. Analytical studies require consideration of an enormous amount of investment scenarios. Such analysis is based on the asset and time "slicing" of a real portfolio available in the past or hypothetical portfolio to be built in the future. Each slice represents some combination of assets and periods. This is done both for the reporting and analytical purposes. However, mathematical methods in its present form need all transactional information within the chosen slice as an input. If data is changed, say the weight of some asset changes slightly, all calculations to be redone again from scratch.

Geometric linking might be helpful in certain restrictive situations in producing an approximate value for the rate of return, but there is no way to control the accuracy of this estimate. The same consideration is true for the TWRR. Nonetheless, geometric linking is used because there were no alternatives to it until recently. Otherwise, analysts have to stay with laborious recalculations for every new slice they create or for any change in the transactional data. This is the reason why in most PM (investment performance measurements) systems intermediate results for asset-period combinations are stored in the databases. Such approach accordingly leads to the huge databases and complicated performance management and analytical systems. The name "Universe" used to name these databases is really a meaningful one. Analysts are restricted in their choice of possible investment scenarios. Results are also approximate ones when using TWRR, or geometric linking to link returns found by other methods. There is no way to estimate how far the results are from the actual return (what we proved is MWRR). We note that Modified Dietz estimation generally is a more accurate one than the geometric linking. The last one has such a high volatility for the real portfolios that its usage becomes questionable. A more detailed consideration with numerical examples can be found in (I. Chestopalov 2004).

Presently, the overwhelming majority of analysts and fund managers are in need of the following functionalities:

- creating arbitrary combinations of assets and periods and calculating their rates of return in real time or close to a real time fashion.
- apply some optimization criteria toward a wide variety of assets so that the system should be able to find the optimal composition of assets and periods.
- monitoring the market value of portfolios in a real time.

The list can include many more functions presently not supported due to the restricted nature of existing mathematical approaches.

The correct calculation of rate of return affects other areas of performance measurement business, such as attribution analysis and risk adjusted rate of return. Correct value for the rate of return makes a big difference in these applications. This is becoming increasingly important today, when the ever-more sophisticated risk assessments are based on not only rate of return itself, but also its derivatives. So, an error embedded into the rate of return will migrate to other parameters used for the risk evaluation, thus jeopardizing the results of risk analysis.

In summary, the following problems impede the industry progress in the system development and analytical research.

Analytical research

- restrictive analytical scenarios and insufficient choice of analytical tools available to
- analysts
- slow performance of systems used by analysts in the research process
- computationally intensive and time consuming methods
- uncontrolled accuracy of calculations in many instances, including risk valuation

System design

- staggering redundancy of computations
- huge database size
- unsatisfactory data structure
- complicated and big systems, that are difficult to develop, deploy and support
- even if a bit of data is changed, then *all* returns must be recalculated where the data
- were previously used

Shestopaloff's linking methods improve the situation dramatically

Problems listed above are addressed by new mathematical methods collectively called Shestopaloff's linking methods (SL) (A. Chestopalov *et. al.,* 2005). The idea behind these methods is similar to geometric linking. However, unlike geometric linking Shestopaloff's linking produces a correct value of rate of return for Modified Dietz formula and approximate value with predefined accuracy for MWRR. Also, SL methods allow linking *asset* returns for the same period while no other method including geometric linking can perform this operation. This is an extremely important feature never available in the investment performance measurement business before.

It was mentioned already that generally the equation defining MWRR can be solved only numerically. Thus the return for MWRR is an approximate one. However, it can be computed with any required accuracy. The same is true for SL. Consideration about controlled predefined accuracy is valid for Shestopaloff's linking of MWRR found for combination of periods and / or assets. That is, the result is also an approximate one, but it can be computed with required accuracy. For comparison, when Modified Dietz is used for the evaluation of TWRR, or geometric linking is used for finding rate of return, the error is unknown and there is no way to evaluate it.

Another distinguishing feature of SL is its ability to perform linking not only across periods and asset combinations, but also simultaneously for the assets and periods. This is a new functionality that never existed before. It opens a whole new area of analytical research. This feature has no present analogies and is extremely important for the day-to-day business operations, accounting purposes, investment research and optimization of investment strategies. Because of the importance let us to reiterate the said above, that is: geometric linking cannot be used to link returns across assets. SL is the only mathematical approach capable to do this.

The Shestopaloff's linking approach is also an extremely valuable entity from a system design and data management perspective. It allows dramatically reduce the size and complexity of performance management systems. This accordingly cuts development, deployment, training costs and required time to a fraction of previously allocated resources. Application of SL methods also boosts system performance hundred and thousand times moving computations actually to a real time domain.

Let consider how the Shestopaloff's linking methods solve current industry problems. Please note that all numerical estimations below are done on a very conservative and reasonable basis, not the optimistic one. These numbers outline the bottom line; the actual gains will be higher.

Analytical scenarios.
Rate of return for each asset-period combination with SL approach is calculated only once. In addition to rate of return three other numbers are calculated at the same time for this atomic combination. After that this asset-period set of four numbers can be used in any combination with other asset-period slices. These four numbers are the only data representing and characterizing these slice for the outer world, though it may contain thousands and thousands transactions and all kinds of financial instruments. This way such slice can be included into *any* combination of assets and periods. Calculation of the total return is based on these four numbers regardless how many asset-period slices compose the final combination. This collapsed amount of data is the reason why the speed of computations becomes very close to a real time for the all practically meaningful scenarios. Instead of rearranging data for thousands and thousands transactions every time we use only four numbers, always. So, analysts are getting accuracy and objectivity of results with SL methods, enormous flexibility of their research and reporting and actually a real time system performance.

Abstraction from the transactional data layer brings even more flexibility and new analytical approaches. From now on, investment research can be coupled with mathematical optimization tools due to the independency of each atomic asset-period combination return on the neighboring data. Optimal combination can be found automatically and quickly by using available optimization tools successfully used in other areas for a long time. Actually unlimited number of investment scenarios and strategies can be analyzed automatically and very quickly: within the reasonable restrictions in a real time.

Performance.
Example. Calculating the rate of return for 1,000,000 periods with SL methods takes 0.75 sec on desktop computer, while calculation of the same number of MWRR rates of return requires 192 sec on the same computer. That roughly corresponds to calculating rate of return for 3,000 years, based on daily returns. This proves that even on a desktop computer SL provides real time computation of rate of return. A way to further increase computational speed is to use the inherent capability of SL methods to utilize available computational power completely through the multithreaded and multiprocessor

application designs. SL parallel computing can be done naturally, without any additional costs, while present methods require all data to be available within a single processor as a monolithic chunk of data. So, systems' performance gain in the production environment will be thousands times and more.

Accuracy.

SL methods produce precise value for Modified Dietz method. As for the money-weighted rate of return, SL provides approximate value for MWRR with predefined accuracy – same as the present calculation of MWRR does.

Redundancy of Computations.

There is no computation redundancy when using Shestopaloff's linking methods. With existing methods adding new asset to the same period and / or changing period requires complete recalculation of such sub-portfolio, despite the fact that the rate of return for a previous sub-portfolio has been calculated already. However, there is no way to reuse this information within Modified Dietz or MWRR approaches even if a minor change was done. Shestopaloff's linking removes this kind of redundancy *completely*.

Database size.

There is no need to store intermediate results with Shestopaloff's linking methods. Only returns for the atomic combinations of periods need to be stored. Rate of return for any other composition of assets and periods is calculated on the fly. In addition, Shestopaloff's allows doing cumulative calculations. That is, it allows adding the return for a new period to some previously calculated return for a longer period. For example, adding a new daily return to the return during the past few weeks, calculated previously, is actually an instant operation. That further boosts performance and reduces computational time. Compared to the present applications, database size of SL based application is expected to be at least tens times smaller, if not hundreds. Hence, development, deployment and maintenance costs become miniscule compared to the present situation.

Data structure.

Data management and data structuring is a big issue in investment performance management business and accordingly in system design. A few of the reports presented at 2006 PMAR Conference emphasized this problem. The data structure with Shestopaloff's linking becomes logically and physically clear and transparent. The set of transactional data is separated from the calculated rates of return. The optional third data abstraction layer can be temporary in-memory or other storage of returns that were calculated on the fly for the analytical research and report generation. Research, optimization and analytical tools can be connected to this set of data on a session basis. When the job is done, these data can be removed, because they can be readily reproduced due to the real time system performance. This approach might be needed when some system super-performance is required. It can be the case of huge analytical systems processing an enormous amount of data. Overwhelming majority of performance management systems will provide close to a real time system performance without this temporary data layer anyways.

Thus all data are structured with Shestopaloff's linking into three logically and functionally separated sets of data. Data consistency becomes extremely high because there is no data duplication and data interdependency at all. From the system design perspective this is a feature that is impossible to over-appreciate.

Figure 3 shows data separation achieved through the application of Shestopaloff's linking methods.

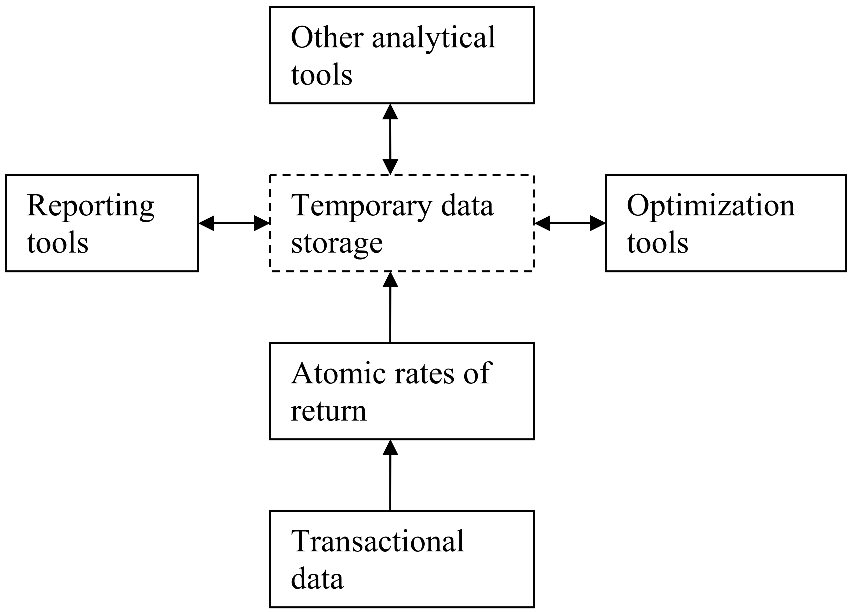

Figure 3. Structure of Performance Management Data Achieved with Shestopaloff's linking Methods.

System design.
In addition to the database design and data structuring, the application itself becomes small and compact. This is because currently the main bulk of code and application modules in PM systems handle complex assemblies of transactional data into asset-period slices. The code surmounts also because of the following storage and retrieval of intermediate results. SL algorithms do not require any original data assemblies into the whole asset-period composition before computations, unlike presently used methods.

Another complication in the current designs deals with queues of batch jobs, monitoring of jobs, retrieval of the results, etc. Job queues can be eliminated as such because of the high application performance and possibility of efficient multithreaded and multiprocessor system designs with SL. Operating systems themselves have built in optimized low level mechanisms to handle such multiple requests. The overall system design becomes extremely efficient and requires significantly less resources for the system development, deployment and support with Shestopaloff's linking methods. Development time can be reduced dramatically as well.

Data change.
If data change occurs, only four integral values for the atomic asset-period combination have to be recalculated with SL approach. The earlier calculated returns will remain untouched.

Conclusion

Development and evolvement of any entity is a natural phenomenon, this is the way human activity progresses. The performance measurement business experienced lots of new developments in recent years. However, the real breakthrough in the fundamentals was not achieved since Dietz introduced his formulae. Computers contributed a lot toward moving in the direction of money-weighted rate of return. Nonetheless, the idea that MWRR is the only true rate of return and the rest is nothing more than its approximate derivatives still requires recognition by the general public.

Shestopaloff's linking methods represent a real qualitative advance in the mathematical approaches used in this industry, given the efficiency of solutions they provide for the long term problems this industry has been struggling with for decades. The Shestopaloff's linking concept can eventually move the whole industry to a new level; make it up-to-date and compatible with mathematical methods of optimization and research analytical tools widely used in other industries. There is an unfilled gap in the mathematical methods currently circulating within the industry. It impedes further progress toward more advanced and efficient approaches and tools capable of moving the industry smoothly to the next qualitative level it should be on, given the advances in technology and knowledge base achieved in other industries. Performance measurement is a conservative business and by its nature it has to be. However, the actual situation is such that Aristotle's "golden mean" will not be broken but rather will be improved if the MWRR becomes a standard. Shestopaloff's linking should also receive comprehensive, thorough and objective consideration by the financial community. Eventually the whole industry will benefit tremendously if it does not ignore its enormous benefits hidden by a light cover of non-traditional thinking.

REFERENCES

Spaulding, David, "Measuring Investment Performance", McGraw-Hill 1997.

Spaulding, David, "Investment Performance Attribution", McGraw-Hill 2003.

Feibel, Bruce, "Investment Performance Management", John Wiley & Sons 2003.

Global Investment Performance Standards (GIPS™). As Revised by the Investment Performance Council 7 December 2004 and Adopted by the CFA Institute Board of Governors 4 February 2005.

Chestopalov Alexandre and Konstantin Chestopalov, "Consistent linking Concept for Fast Calculation of Rate of Return and Research of Investment Strategies," *The Journal of Performance Measurement* , Fall 2005, Vol. 10 No. 1, pp. 50-63.

Illmer Stefan, Wolfgang Marty, "Decomposing the Money-Weighted Rate of Return," *The Journal of Performance Measurement,* Summer 2003, Vol. 7 No. 3, pp. 42-50.

Chestopalov Iourii and Sergei Beliaev, "A Simplified Method for Calculating the Money-weighted Rate of Return," *The Journal of Performance Measurement,* Vol. 9 No. 2, pp. 14-23.